Catalytic Representations in Mathematics

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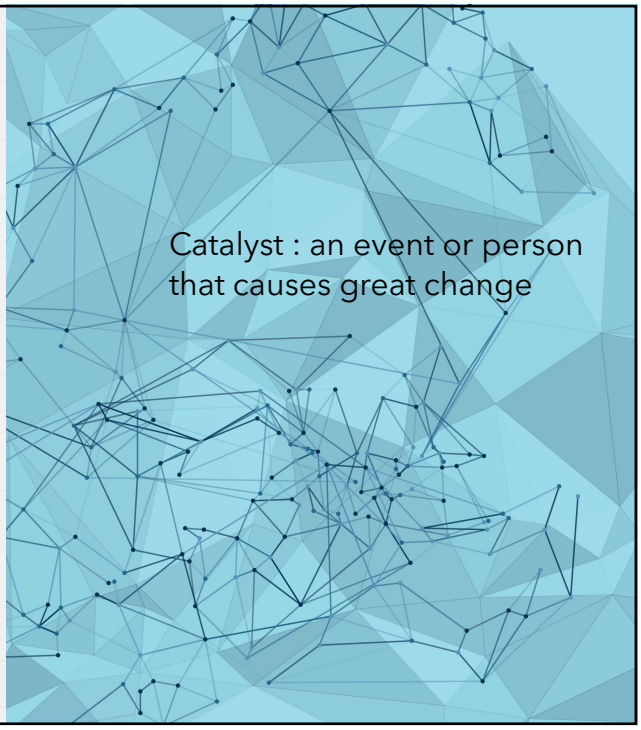
University of Sydney

2024 CMA Conference
Mathematicians: Agents of Change



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Representation Theory

Mathematics is a collection of abstract ideas organised by humans into systems to describe patterns, relationships and structures we encounter in the 'real' world or hypothesise to exist.

- Representations are created by humans to make abstract ideas more tangible and easier to work with.

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Think of various ways to represent the concept of 'half'

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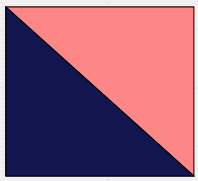
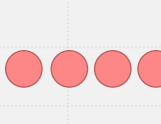
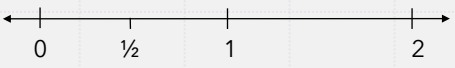
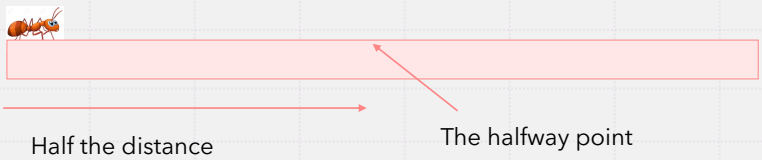
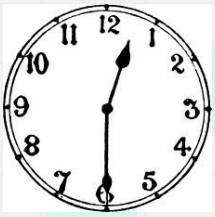
$\frac{1}{2}$











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One mathematics concept → Many representations

The fact that there are so many representations for 'half' demonstrates that each representation is not the actual mathematics concept.

- Different representations can highlight different characteristics of a concept and/or differences in meaning in different contexts.
  - Multiple representations of one concept help us to deduce what the concept is and gain a deeper understanding of it. We must 'see through' the representation to what it represents.
    - The 'overlap' between representations allows generalisation and abstraction.

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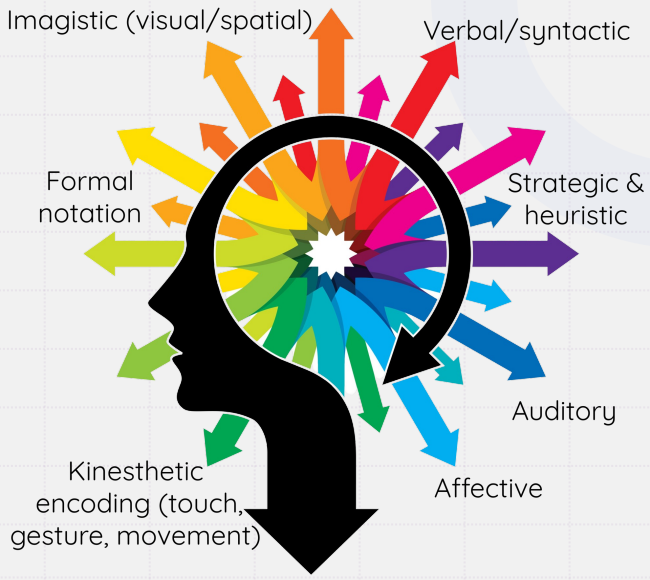
Interconnected representation systems in the brain

Neuroscience: Different parts of the brain are activated when different representations are processed

Internal representations

Cognition theory



Goldin, G. & Shteingold, N. (2001).

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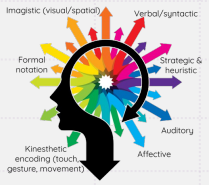
Internal Representations

Constantly adapted and interconnected systems:
- Verbal/syntactic
- Imagistic
- Auditory
- Formal notation
- Strategic (problem solving)
- Affective
- Kinesthetic visualisations



Acts of creation & interpretation

The focus of teaching is supporting meaningful development of representation systems. Incomplete representation systems make barriers to learning.
(Goldin & Shteingold, 2001)

External Representations

For example:
- Concrete materials and models
- Pictures and diagrams
- Movement and physical experiences
- Symbols, equations and written language
- Hand drawings
- Digital representations

Self-created or pre-created by someone else

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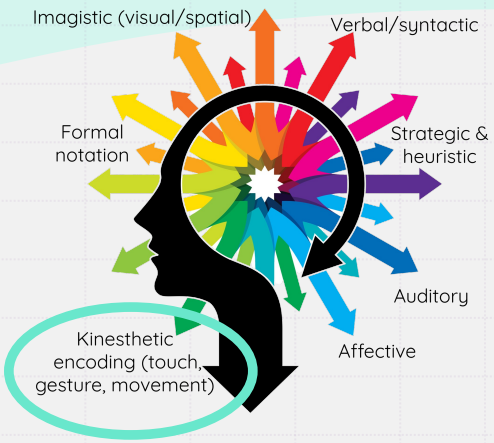
Embodied modes of representation

- The origins of embodied representation lay in the movement of the body itself and/or in the interactions between the body and the external environment (Hutto et al., 2015)
- Embodied approaches utilise biologically primary functions and sensorimotor learning to attend to and enact mathematical concepts and processes.



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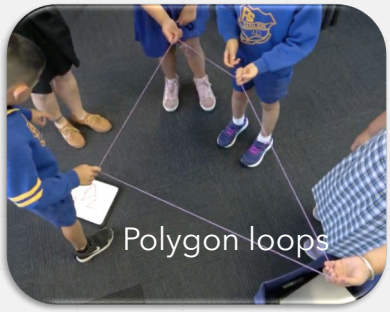
Embodied Learning in Early Mathematics and Science Project
2022-2024

Embodied Learning in Early Mathematics & Science

- Purposeful (concept-linked) use of gesture, pointing, touch, tracing, body movement and drawing by both teachers and students improves student engagement, communication and learning outcomes.



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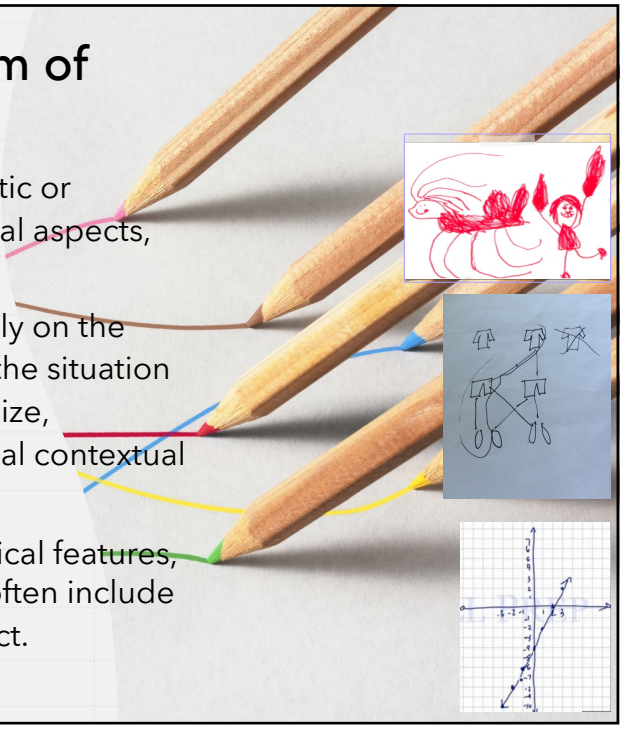
Drawing is an important form of representation

- **Natural drawing** is more playful and artistic or pictorial (but still may contain mathematical aspects, like perspective).
- **Mathematical drawings** focus deliberately on the mathematical concepts and processes in the situation being represented (e.g., number, shape, size, sequence etc.), but often contain additional contextual features.
- **Diagrams** contain only the key mathematical features, spatial structures, and relationships, and often include symbols (e.g., arrows), so are more abstract.



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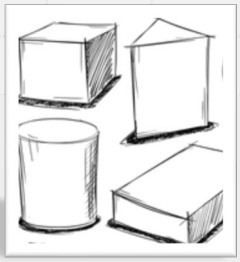
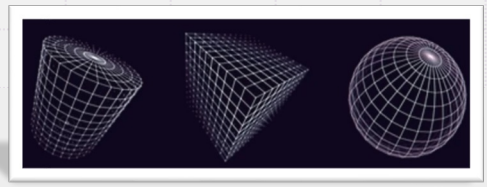
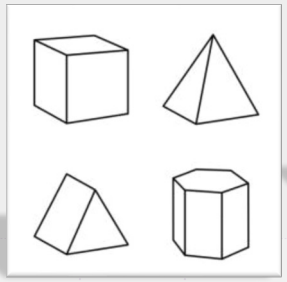
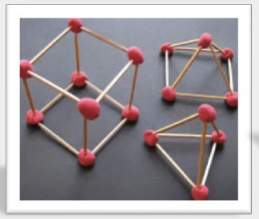
Different representations highlight different features or properties

Making explicit connection between representations, including verbal, strengthens conceptual understanding

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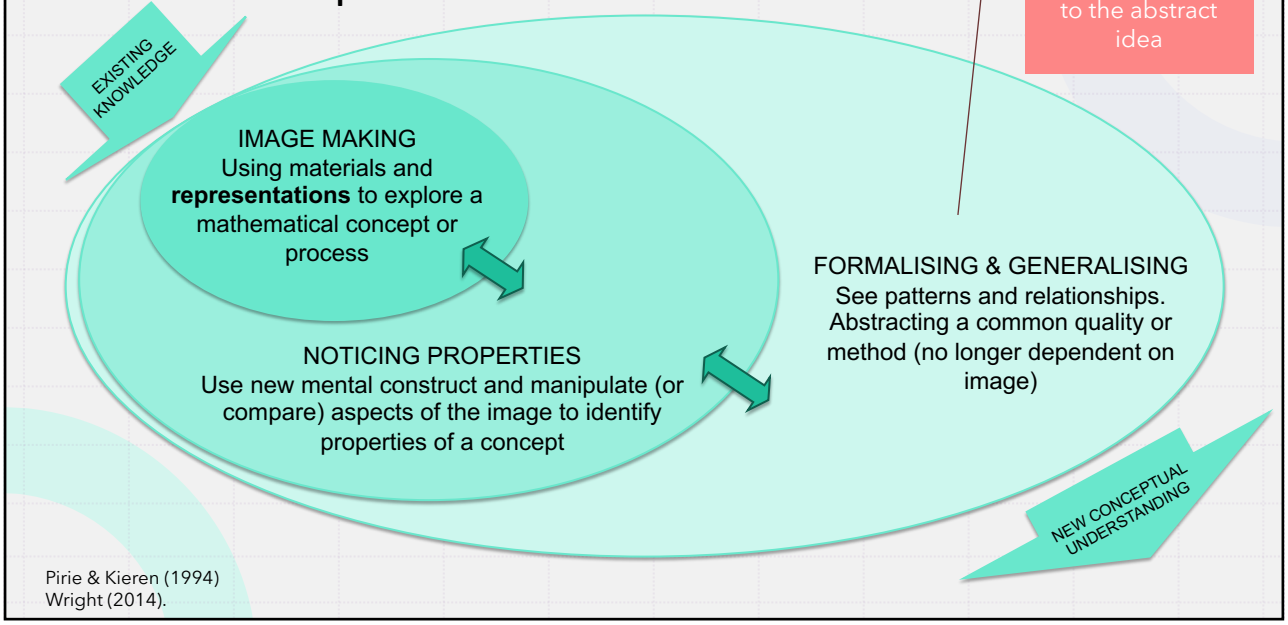
Learning theory that helps make sense of the role of representations



Pirie & Kieren (1994)
Wright (2014).

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Solve this problem. Consider how to convince someone else of your solution.

I got on the lift and went up 6 floors.

I then went down 8 floors.

Then up again 3 floors, where I got off the onto the 5th floor.

On which floor did I first enter the lift?

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- In solving the problem did you use representation to initially explore the problem?
- Did you switch to a different representation during the process?
- What representation would you use to present your solution?

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Representations can be thinking tools or communication tools

This means that:

- The act of representation (verb) can lead to the creation of new knowledge (i.e. learning).
- We also use self-created representations to support 'working memory'. (e.g., gesture, sketches, listing)
- A representation (noun) can communicate completed thinking or methods. It presents a summary.

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Representations and problem solving

"… it is well known that the "difficulty level" of a given task can be changed by years – simply by changing the context or the **representational media** in which problems are posed (e.g., written symbols, written language, diagrams or graphs, concrete models, or experience-based metaphors)."

Lesh, English, Riggs, & Sevis (2013)

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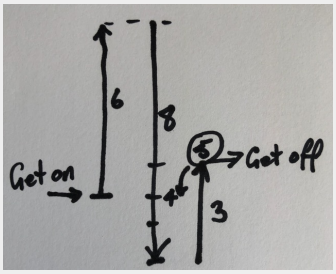
How do these representations relate to each other?

Act it out

Enactive



Draw a diagram

Iconic

$? + 6 - 8 + 3 = 5$

$? + 1 = 5$

$? = 4$

Create equations

Symbolic

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Representational fluency

- Being able to switch between representations allows a problem to be explored in different ways, and more effective strategies can be selected.
- Telling students what representation to use in solving a problem may not be helpful if it is disconnected from their own repertoire.
- Supporting the development of a range of representations and the relationships between them is likely to improve 'sense-making' skills and boost problem-solving success.

Heinze et., al (2009)

This brings us back to self-created vs pre-created representations

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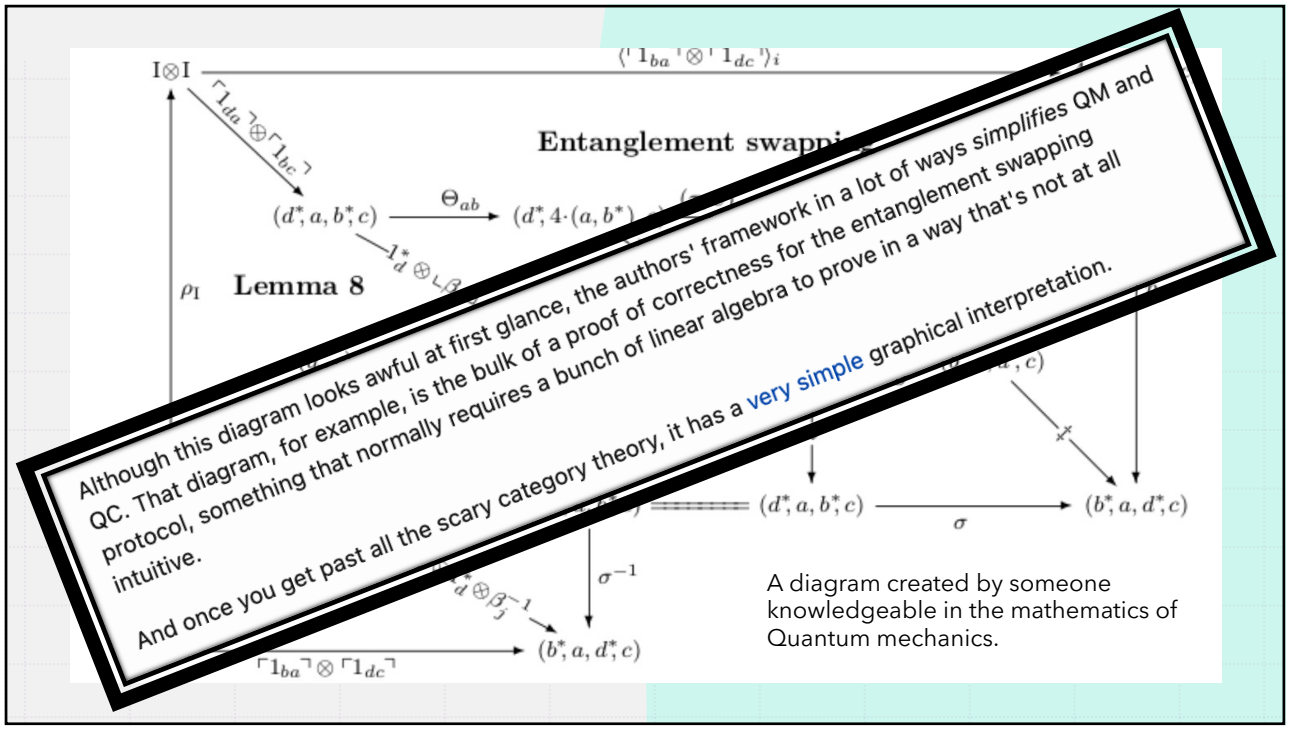

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1. Use a mental strategy to work out **7 + 15**
2. Think of another way to do it.
3. Draw a **number line** to represent each strategy.
4. What do you think your students would draw?

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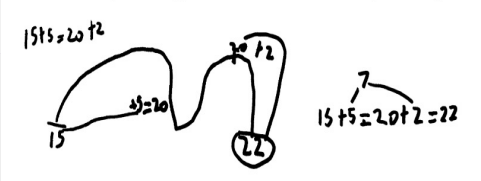
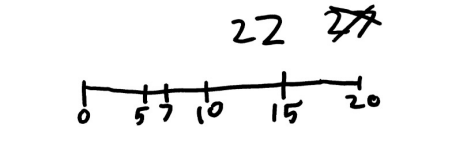
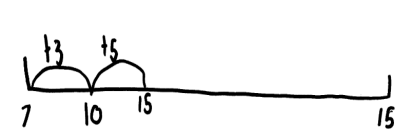
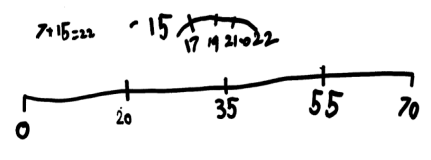
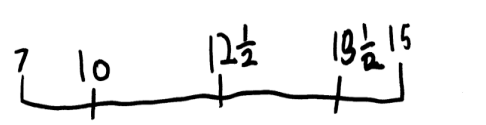
Year 3 students' responses to 7 + 15

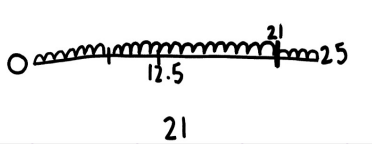












Way & Harding (2017)

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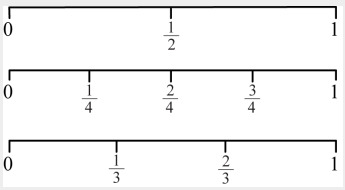
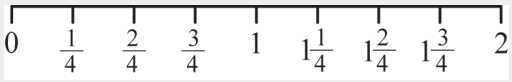
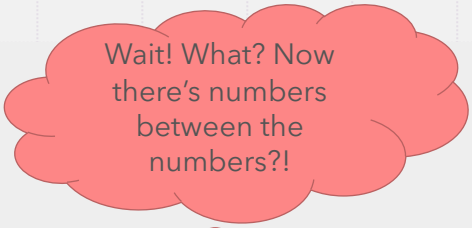
Fractions on a number line





Stage 2 in NSW Syllabus (2012)



The new K-6 syllabus places much greater emphasis on the development of understanding of diagrammatic representations

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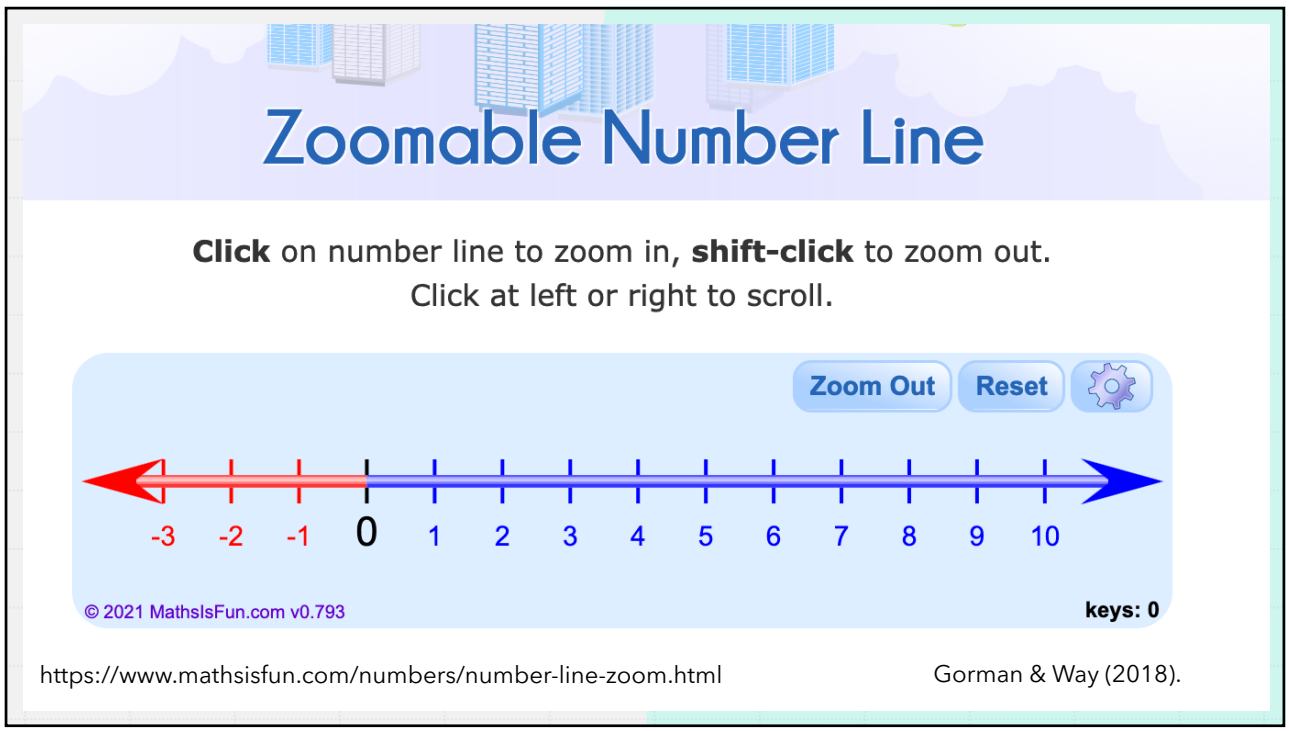

https://www.mathsisfun.com/numbers/number-line-zoom.html Gorman & Way (2018).

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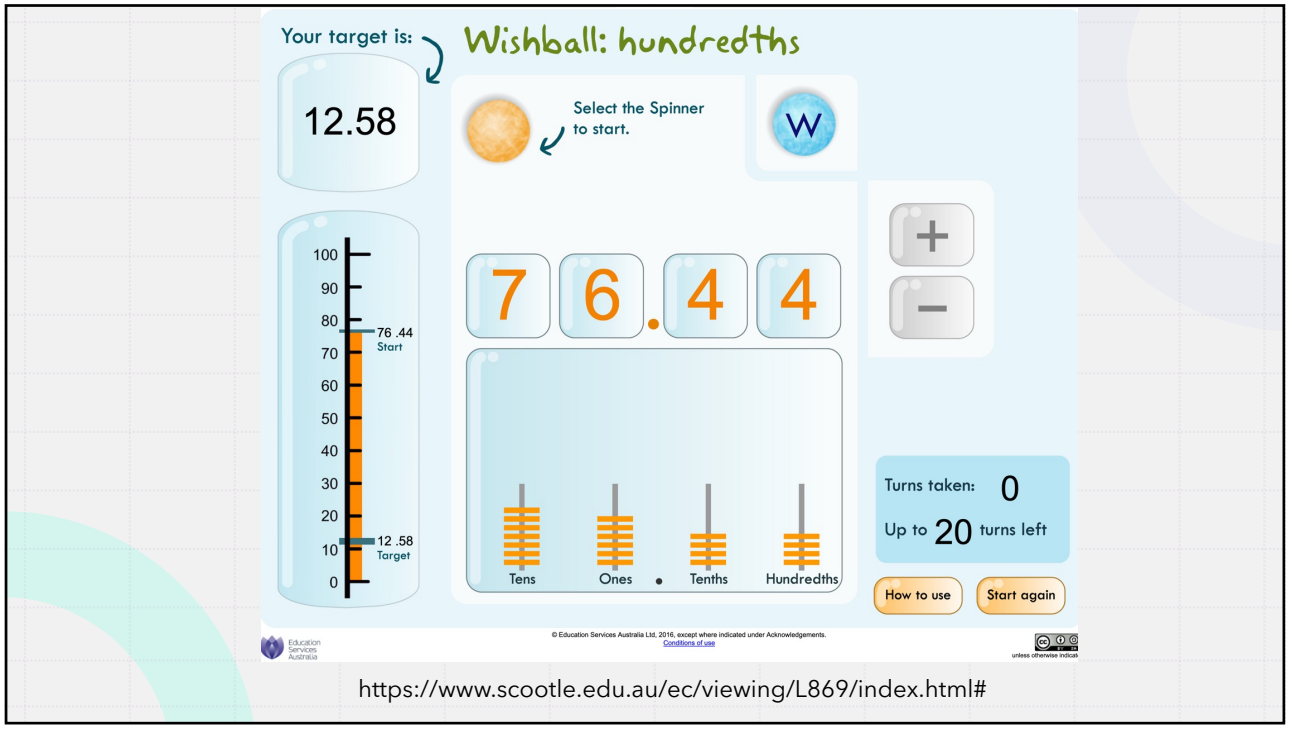

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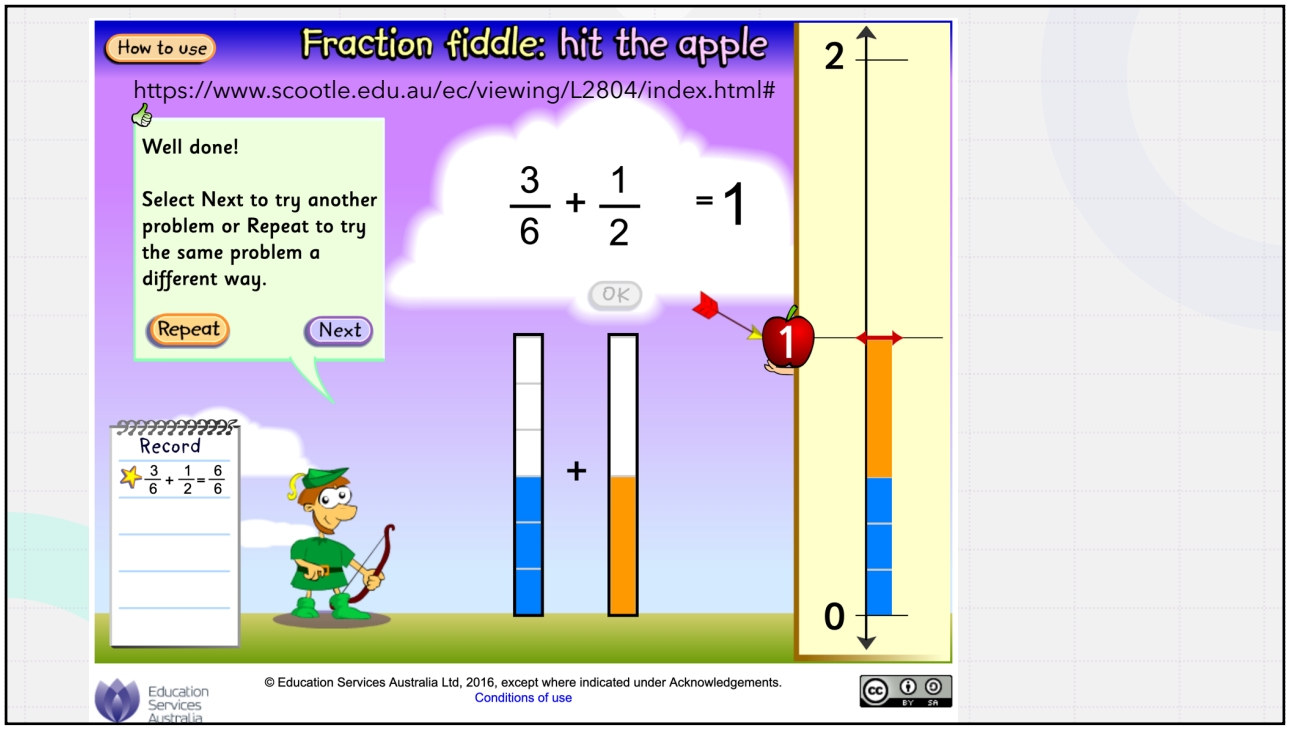

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Key messages

- An important role for teachers is to select representations of mathematics that will act as catalysts for shifts in understanding in the students. (Way, e.a. 2015)
- The interplay between external representations and internal representation systems is the main target of teaching.
- Embodied learning approaches offer powerful modes of representation that enhance learning.
- Students' mathematical drawing skills need to be explicitly coached and developed as both thinking tools and communication tools.
- Conventional mathematical diagrams and symbols should be considered a goal of learning rather than the starting point, and students' self-created representations viewed as a pathway of understanding.

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