SHORT CIRCUIT

Canberra Mathematical Association Inc.

VOLUME 15 NUMBER 1 JANUARY 2024

NEWS AND COMMENT

Short Circuit wishes you a happy and rewarding New Year, 2024.

Stay tuned for a new Welcome Event, about to be restored after the Covid crisis. Details will be announced through this mailing list.

Memberships can now be renewed for 2024. Forms are available on the CMA website. Membership fees for 2024 are $95 for individual members, $150 for institutional members, and one year of free membership for full-time students (final year encouraged).

MEMBERSHIP

Memberships run from 1 Jan to 31 Dec. each year. Membership forms may be downloaded from the CMA website:
http://www.canberramaths.org.au

The several benefits of Membership of CMA may be found on the website.

NEWSLETTER

The CMA newsletter, Short Circuit, is distributed monthly to everyone on our mailing list, free of charge and regardless of membership status.

That you are receiving Short Circuit does not imply that you are a current CMA member but we do encourage you to join.

Short Circuit welcomes all readers.

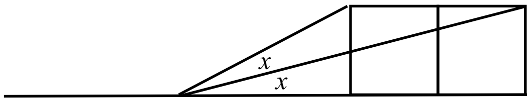
CANBERRA MATHEMATICAL ASSOCIATION

Inside:

PUZZLES

1. Hard way, easy way

In the diagram below there are two adjacent identical squares and some oblique lines that form angles x and $2x$ with the base line. How big is $2x$?



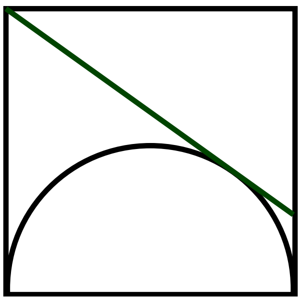
2. Ramanujan's Christmas tree

$$3$$
$$= 1+\sqrt{4}$$
$$= 1+\sqrt{1+\sqrt{9}}$$
$$= 1+\sqrt{1+\sqrt{1+2\sqrt{16}}}$$
$$= 1+\sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{25}}}}$$
$$= 1+\sqrt{1+\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}}$$

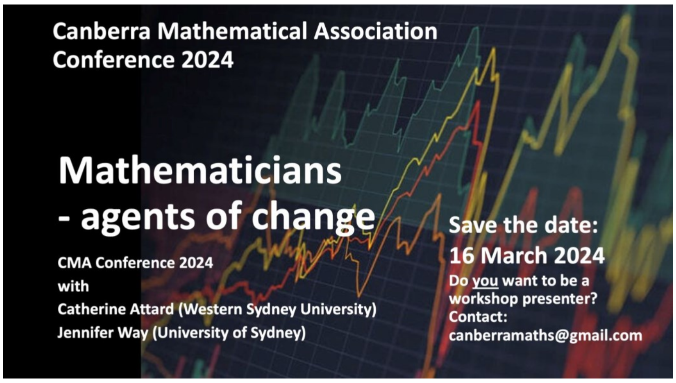
How far does the pattern continue?

3. A bit less than √2

In the following diagram, a semicircle is inscribed in a unit square. Find the length of the line segment tangent to the semicircle reaching from a corner of the square to a side.



CMA 2024



IM^{2}C 2024

Registrations for the International Mathematical Modeling Challenge 2024 are open.

The IM2C 2024 will occur from 13 February – 26 March 2024.

For more information, visit the **IM2C** website.

MAWA VIRTUAL

Join us for the 2024 Virtual Maths Conference on Friday, 3rd May.

ICME-15

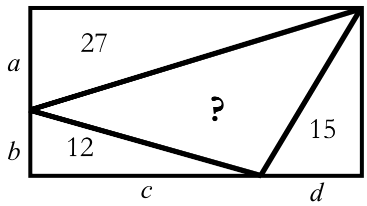
The International Congress on Mathematical Education is the largest international conference on mathematics education in the world.

The 15th International Congress on Mathematical Education (ICME-15) will take place 7-14 July 2024 at International Convention Centre in Sydney, Australia. ICME-15 promises to be an innovative congress that builds on the well-established ICME program, showcasing established and emerging thought leaders from around the world.

PUZZLE SOLUTIONS from Vol 14 No 12

1. Dissected rectangle

The numbers in the triangles represent areas.

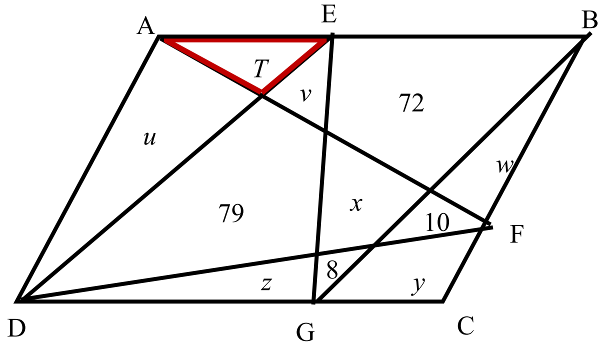


The parts of the vertical and horizontal segments have been labelled *a, b, c, d*. The area we seek is $(a + b)(c + d) – 54$.

We have equations $a(c + d)/2 = 27$, $bc/2 = 12$, and $d(a + b)/2 = 15$. These three are not enough to determine the four quantities *a, b, c, d* uniquely. Indeed, if the height of the rectangle were to be doubled and the width halved, all the triangles would keep their stated areas. Thus, *any* solution satisfying the three equations should lead to the missing area.

Taking $d = 3$, for example, and working through the algebra, we find $(a, b, c, d) = (6, 4, 6, 3)$ is a solution. Thus, the area of the missing triangle is 36.

2. Dissected parallelogram



We labelled the unknown areas *u, v, w, x, y, z* and *T*.

Consider the sets of triangles with bases making the sides of the parallelogram and vertices on the opposite sides. We see that each such set is half the areaof the parallelogram. Therefore,

$T + v + 72 + w + y + 8 + z = u + 79 + x + 10$ and
$z + 79 + v + y + 10 + w = T + u + 72 + x + 8$.

That is, $T – u + v + w – x + y + z = 9$ and
$T + u – v – w + x – y – z = 9$.

Adding these gives $2T = 18$ and so, $T = 9$.

3. One over the radio

Two pirates, at night, wish to divide a cache of approximately 300 gold coins equally between them but there is one coin left over. So, they wake up another pirate, thinking of a three-way split. Again, there is a coin left over. In turn, they wake up a fourth, fifth and sixth pirate and in each case there is a coin left over. When a seventh pirate is brought in, an equal division is finally achieved.

How many coins were there?

We look for a number near 300 that leaves a remainder of 1 when divided by 2, 3, 4, 5, or 6, but which is exactly divisible by 7. By guess-and-check the answer 301 is easy to find. This is 43×7.

Is there a smaller solution?

Smaller multiples than 43 that leave a remainder of 1 when divided by 6 are 7, 13, 19, 25, 31, 37. In modular arithmetic, 7 is equivalent to –1. So, in modulo 4 its multiplier must also be –1. Thus, we can eliminate 13, 25, and 37 from the list. Similarly, $7 \equiv 2 \pmod 5$. So its multiplier must be 3 in mod 5 because $2 \times 3 \equiv 1 \pmod 5$. The smallest number not yet eliminated from the list that has this property is 43. So, 301 is the smallest solution.

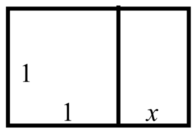
Is there a larger solution than the one you found?

$301 \times 301 = 90601$ is a larger solution.

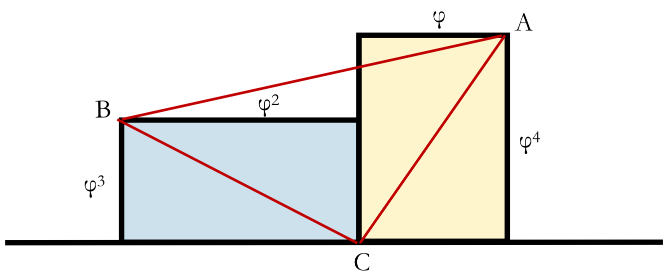
Are there infinitely many numbers with the same divisibility property?

Given any two solutions, their product is also a solution. So, infinitely many numbers have the property.

4. Golden ratio



It is specified in the diagram that $(1+x)/1 = 1/x$.. Equivalently, $x^2 – x – 1 = 0$, with solution $x = (\sqrt{5} – 1)/2$. We write $1+x = \varphi = (\sqrt{5} + 1)/2$.



By Pythagoras BC = AC. The triangle is isosceles with sides φ^5, φ^5 and $3\varphi^4 + \varphi^3$. (The original diagram was drawn less accurately.)

NEWSLETTER OF THE CANBERRA MATHEMATICAL ASSOCIATION INC. INC

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We're on the Web!
http://www.canberramaths.org.au/

ABOUT THE CMA

The Canberra Mathematical Association (Inc.) is the representative body of professional educators of mathematics in Canberra, Australia.
It was established by, among others, the late Professor Bernhard Neumann in 1963. It continues to run - as it began - purely on a volunteer basis.

Its aims include

- the promotion of mathematical education to government through lobbying,
- the development, application and dissemination of mathematical knowledge within Canberra through in-service opportunities, and
- facilitating effective cooperation and collaboration between mathematics teachers and their colleagues in Canberra.

THE 2024 CMA COMMITTEE

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Theresa Shellshear is CMA's COACTEA representative.

Bruce Ferrington is CMA's AAMT representative.

Joe Williams is the website manager.

Short Circuit is edited by Paul Turner. **ISSN 2207-5755**

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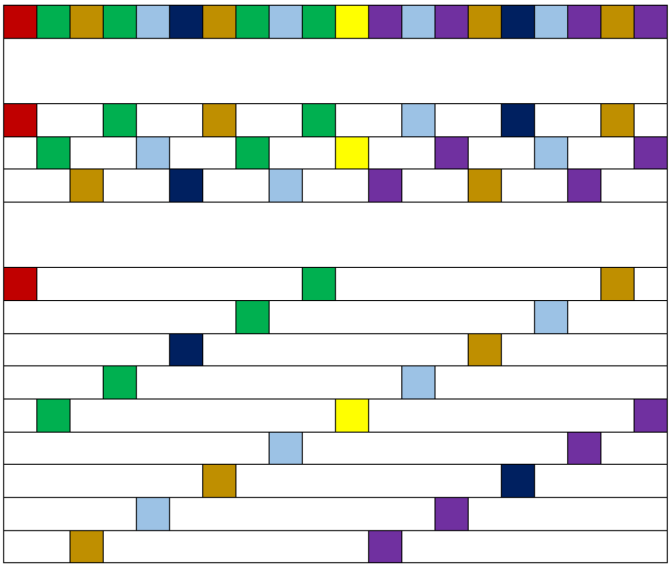
SELF-SIMILAR DESIGNS



The top strip of coloured blocks in this diagram has the property that when expanded by a factor of three, preserving the order of the colours, the new positions of the blocks correspond with blocks of the same colours in the original strip. Again, if this second strip is expanded by a factor of three, the same correspondences occur. Imagine 9 of the first strip set against 3 of the stretched-out second strip and 1 of the stretched-out third strip.

Here is another (different) example, expressed in typographic symbols:

□ × • × ° \ • \ □ × • × ° \ • \ □ × • × ° \ • \
□ × • × ° \ • \

The top row gives the basic sequence of symbols three times while the second row goes through the sequence once. The distances have been expanded uniformly, yet the symbols correspond vertically.

These objects are like their fractal analogues in two -dimensional space, but they belong to a line. The line might be measured in physical distance or as durations in time or there might be some other line-like model.

The coloured boxes with which this article began are in fact a representation of a musical process used by Tom Johnson, a venerable American experimental composer, in his piece *La Vie est si courte*. The idea is that the melodic line will be played or sung at three different tempos simultaneously yet there will be no harmonic clashes. Johnson's explanation with musical illustrations is well worth watching.

For the next part of this article, I reconstruct the mathematical thinking behind Johnson's design.

There are twenty boxes in the sequence. Thinking of them as durations measured from a starting point we can number the left-hand edges from 0 to 19. Also, the repeating nature of the pattern suggests that modulo twenty arithmetic will be appropriate.

The box at position 1 is green. If the sequence is expanded by a factor of 3, the second strip of blocks will have a green block at position 3. So, a matching green block at position 3 in the first strip is required. Then, a green block at position $3 \times 3 = 9$ is needed, and another at position $3 \times 9 = 27$ which, in modulo 20, is position 7. Then, $3 \times 7 = 21$, which is 1 in mod 20, so an orbit has been completed comprising positions 1, 3, 9, 7. This corresponds to the positions of the green boxes in the top strip.

The structure with numbers 1, 3, 7, 9 under multiplication modulo 20 is a *group*. Recall that the element 1 belongs to every multiplicative group, and for every element there is an element such that their product is 1. For example $3 \times 7 = 1$ in mod 20. It turns out that at the heart of any self-similar design there is a multiplicative group structure.

To assign colours to the remaining 16 boxes in a way that maintains the correct distance relationships, the group elements multiply in turn each of the remaining elements in the set of numbers 0 to 19.

Each group element times zero is 0. So there is just one red box. The group multiplied by 2 gives the set 2, 6, 14, 18, corresponding to the brown boxes. Continuing in this way we obtain the orbits

0	red
1, 3, 7, 9	green
2, 6, 14, 18	brown
4, 12, 8, 16	blue
5, 15	dark blue
10	yellow
11, 13, 17, 19	purple

Since there are seven orbits, there can be seven colours or, in Johnson's musical version, seven pitches. Johnson considers five different notes to be enough for his melody, so he assigns silence to two of the orbits. The silences are gaps in the melodic line that create a pleasing rhythm.

The result is reproduced below in musical notation showing the composer's assignments of orbits to musical pitches.

In the second example of a self-similar design given above (the one made from typographic symbols) the repeating loop has length 8 and the magnification factor is again 3.

This time, we can use modulo 8 arithmetic and obtain the multiplicative group with just two elements 1, 3. The list of orbits is then

0
1, 3
2, 6
4
5, 7.

Thus, five symbols can be distributed as in the example with a particular symbol assigned to each orbit.

Experimentation confirms that loop lengths and magnification factors cannot be chosen independently. Suppose, for example, that we proposed to have a loop length of 14 and a magnification factor of 2.

Every element of the multiplicative group is a power (modulo the loop length) of the magnification factor. That is, the magnification factor *generates* the group elements as powers of the generator. There must always be some power of the generator equal to 1. But there is no power of 2 equal to 1 in modulo 14. So, the proposed design is impossible.

If the generator a and the proposed loop length L have a common factor, we would have $a^n \equiv 1(\text{mod } L)$ for some power n, so that L would divide $a^n - 1$, impossible since L and a have a common divisor.

On the other hand, if the loop length is a prime p then we know by Fermat's little theorem that there exists $a^{p-1} \equiv 1(\text{mod } p)$, (with all the powers of a different), so that the generated group will have p–1 elements. These elements belong to one orbit, leaving no room for a reasonable list of orbits.

A middle ground is achieved by choosing a composite loop length with no factors in common with the magnification factor. We could have a loop length of 21 and a generator 2, producing the group 1, 2, 4, 8, 16, 11 under multiplication mod 21, with orbits

0
1, 2, 4, 8, 16, 11,
3, 6, 12
5, 10, 20, 19, 17, 13
7, 14
9, 18, 15

This allows for six colours/pitches/symbols in a design that is self-similar at magnifications by powers of 2.

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