CIRCUIT

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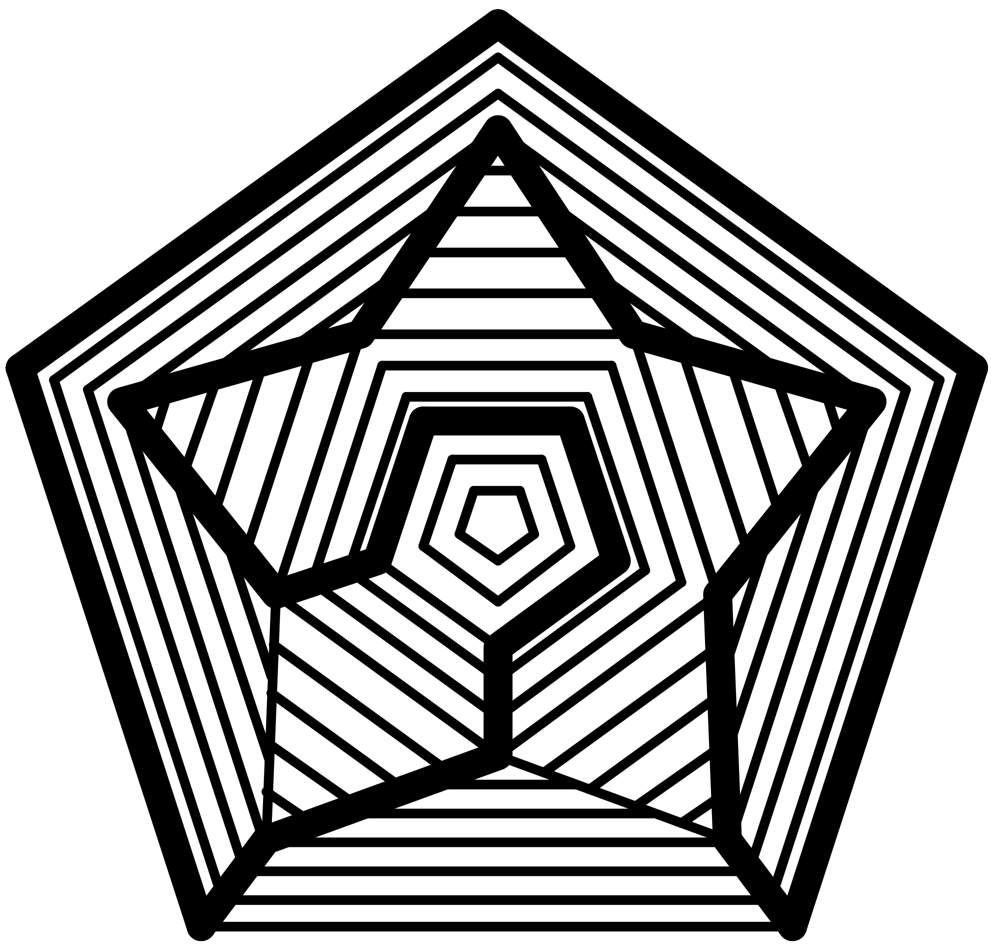


CANBERRA MATHEMATICAL ASSOCIATION

OFFICE BEARERS 2000

President

Paulene Kibble
Numeracy Field Officer
O'Connell Centre
6205 8239 (W) 6241 2673 (H)
paulene.kibble@act.gov.au

Vice Presidents

Margaret Rowlands
Lake Tuggeranong College
6205 6222

Peter Enge
Canberra Institute of Technology
6207 4882
peter.enge@cit.act.edu.au

Secretary

Andy Wardrop
Erindale College
6205 8104 (W) 6288 9739 (H)
axwardro@erindalec.act.edu.au

Treasurer

Steve Thornton
Australian Mathematics Trust
6201 2017
SteveT@amt.canberra.edu.au

Councillors

Bobbie Beale
Canberra College

Peter McIntyre
Australian Defence Force Academy
p-mcintyre@adfa.edu.au

Steve Padgham
Dickson College
josteve-padgham@msn.com.au

Deanne Reynolds
Marist College

EDITOR CIRCUIT *2000*

Peter Enge

CMA POSTAL ADDRESS

P.O. Box 3572
WESTON A.C.T. 2611

CMA INTERNET HOME PAGE

http://www.pa.ash.org.au/canberramaths/

CONTENTS

The objects of the Canberra Mathematical Association are to promote interest in mathematics, to encourage improvements in the teaching of mathematics and its applications, to provide means of communication among teachers and students and to advance the views of the Association on any question affecting the study or teaching of mathematics and its applications.

The Canberra Mathematical Association Logo depicts a Hamiltonian Circuit on a dodecahedron.

CONTRIBUTIONS

Circuit is always keen to receive articles, notes, problems, letters and information of interest to members of the CMA. Please contact the Editor if you wish to contribute to Circuit. If possible contributions should be submitted as Word 97 documents.

FROM THE PRESIDENT

CMA Web Page

Peter Recio has done a marvellous job in setting up and maintaining this site. It has many links to other useful sites and is regularly updated. If you have any queries or want some ideas for your teaching a visit to www.pa.ash.org.au/canberramaths/ is well worthwhile. Thank you, Peter, for your ongoing efforts.

World Mathematical Year

is well underway and our association has contributed in several ways to promoting the role of mathematics in our society.

Thank you to all the members who have contributed to the success of the various undertakings of our association during 2000. In particular thank you to members of the Council who have worked throughout the year to maintain the many functions of the CMA.

ICME-9

Margaret Rowlands and I were fortunate in receiving a grant to attend the 9th International Congress on Mathematical Education in Tokyo in early August. We left the depths of a Canberra winter to do battle with a full blown hot Japanese summer (and a great rail system).

I can't deny how exciting it was to talk in person with luminaries, whose writings you have studied over the years, and to ask the questions that you didn't quite get the answer to in untold hours of study. *What does constructivism look like in the everyday classroom?* A general feeling I left with was that there is still some distance between theory and practice. A year of working in the office reinforced my belief that teaching is an extremely complex enterprise and once removed from a school environment you lose sight of some of this complexity.

We are very fortunate in the ACT, and Australia generally, to be free from some of the extremist views, which have been foisted upon teachers of mathematics in other countries. Pedagogical practices promoted here are at the forefront of world best practice.

Professional Development

There have been several successful PD sessions run this year. These include:

- Alistair McIntosh's workshops on mental computation, which were very popular, with strategies readily applicable in the classroom. Teacher feedback over-whelmingly highlighted a need for support in catering for the range of abilities in the mathematics classroom and Peter Sullivan has been working with teachers today to assist in the process.
- The website workshops run by Steve Padgham and Steve Thornton. These introduced participants to a wide range of web based resources suitable to all school sectors. Again a visit to CMA's site is a great starting point for a journey of exploration.
- Our internet based, interactive voting applet, the development of which was supported by a Discovering Democracy grant, which was presented to teachers at a Discovering Democracy showcase in August by Steve Thornton and me. We also gave a presentation to DETYA representatives in September.

AAMT 2001 Mathematics Shaping Australia

It's at the stage of counting the weeks to go! A small team headed by Beth Lee and Steve Thornton have been beavering away to ensure this will be a Professional Development opportunity too good to miss. The time is right for you to make sure you have registered and offer your services in whatever way possible. Can you chair a presentation? Bring along an idea that has been successful in your teaching? Assist with the one hundred and one jobs that need doing? An email to Beth at ehlee@tpig.com.au would be well received.

Attendance at the Conference may count as part of ACTDECS designated professional development so don't hesitate to include it in your school's PD planning. Thank you to the Department, which has been very generous with financial support to sponsor Marilyn Frankenstein, as well as other Conference expenses.

Canberra Maths Quest

Entries of a high standard, by children from years 5 to 12, were received for our inaugural Canberra Maths Quest. They were presented in a variety of media, which included videos, models and websites. Many thanks to Jacaranda Educational, which is providing awards for section winners.

Discovering Democracy

The voting materials developed by the CMA will be launched at a workshop during the AAMT 2001 Conference.

The Southern Cross Club

We have been fortunate in having this venue to hold meetings throughout the year. I would like to thank the club for making available this space for the many functions we have held here, and add a special thanks to Edie Maier for her assistance with catering.

Some thoughts…

- CMA membership numbers have been maintained around ninety with particular growth in institutional memberships. This has not translated into individuals offering to take on roles in our organisation. The challenge remains to engage more teachers in active membership of the CMA.
- The EBA agreement on professional development supports a whole school or whole faculty approach to PD planning. This has implications for the way we provide PD and offers potential for us to develop different approaches to professional development such as providing PD in stand down.
- AAMT offers a wealth of resources to teachers, which we can more fully promote and utilise.

Thank you to all who have helped me during my time as president. Best wishes to the 2001 Council

Paulene Kibble, 12 October 2000

FROM THE EDITOR

Another end of year looms much too quickly, but it's been a long spell between *Circuits*. My apologies to those contributors who have been waiting months for their articles to appear. At least by the time you are reading this I hope you are winding down, or have already wound down, from the hectic day-to-day routine of working life, and that you now have the chance to relax and reflect for a while.

There are a couple of important threads in this issue and a good range of other pieces to tempt you. The first thread is to do with standardised testing and the second to do with graphics calculators. Both examine the respective issue relative to local and national contexts in Australia. The other piece of note is Chap Sam Lim's thoughtful paper on the role of metaphors in students' learning and understanding of mathematics. It provides rich pickings for any teacher of mathematics. Just the thing for getting minds ticking over in time for AAMT 2001 here in Canberra in January next. (Yes, you can still register!) The

Problems and Activities section has been reduced to a single simple problem, so you won't be too distracted over the Conference.

As usual in the last issue of the year, it is time to thank all those who contributed in any way to *Circuit* this year. Andy Wardrop has been a stalwart with his general encouragement and help with mailout address lists and labels. And for the final time I thank my wife Rosemary for her unselfish and stirling assistance, and her imperturbability during the creation of each issue of the journal.

This is my last Circuit as editor. I have enjoyed the four years at the helm, in spite of the demands on thought and energy, but it is time for me to move over. Peter McIntyre, from the School of Mathematics and Statistics at ADFA, will take over as editor from the beginning of 2001. I wish him well and appeal to all CMA members to make an effort to contribute to the journal. After all, it's your publication.

It only remains for me to wish you all a safe, healthy and happy, festive and holiday season, a flying start to 2001 at the Conference in January, and a good year after that.

Peter Enge

TESTING THE TESTERS

42!

This is the answer to life, the universe and everything, as any readers of Douglas Adams "The Hitchhikers Guide to the Galaxy" will tell you if you don 't already know. Fortunately the book has the words "DON'T PANIC" in large friendly letters on the front cover. "I think the problem, to be quite honest, is that you've never actually known what the question is," explains Deep Thought (the world's most powerful computer). Ministers for statistical education are currently pursuing a self interested and narrow sighted policy of literacy and numeracy testing for our primary students, some even believe there is some worth in publishing statistical tables of "achievement". Here in the ACT we parents are being invited to comment on how "results" might be reported in the future.

Some considerations

What is the nature of these tests?

Here is a short example of one I use to measure the critical intelligence of Ministers for statistical education.

1. What is wholly unique about the number two?
2. What is wholly unique about the number forty?
3. What is the basic mistake made by Deep Thought in calculating 42?
4. What is the probability when two ministers of education meet, given that one of them is female, that both of them are female?

Answers

1. Two is the only even prime number (assumes some knowledge of mathematical ordering).
2. Forty is the only number whose letters are in alphabetical order (assumes knowledge of numbers and the alphabet).
3. 9 multiplied by 6 is equal to 54 (not 42) (assumes a reading of the Hitchhikers series).
4. 1 in 3, when you look at the theoretical possibilities. However the realistic answer is zero if we have only one female minister for education.

Be wary of tests! Here in the ACT the tests have been time consuming and imposed. There is no sense of ownership.

What are the inferences we can draw from the results? Information is essential but the first rule of statistical comparison is to compare like with like; For example, it was argued during the Vietnam War that it was safer for a young Australian to be in Vietnam than to be back in Australia because the number killed in action was much less than the number of young killed on the roads in the comparable period.

To compare one school in a socially disadvantaged area with say a selective school is a statistical nonsense. Schools cannot compensate for society but they can make a vital difference for some of their pupils. School improvement is part of our state machinery for improving human capital.

Real reporting that positively affects a student's outcome comes from student/parent/ teacher dialogue not by political point scoring.

Testing as it is now with the proposed publishing of results is a political distraction. Yes, it might fit in with Mr. Kemp's aim of destroying the ideal of free, popular and public education and yes it is an easier option to drive the middle classes into middle class ghetto private schools, but what does this do for the community?

Averages can be misleading. By definition, an arithmetic average or mean, will always have some values above and some below it, otherwise it would not be an average. And what sort of average will we use. The mean (the sum of all the values divided by the total number in the series) or the median (middle value) or the mode, which is the most frequent number in the series. There will always be AVERAGE schools. This is not to say that these schools are not effective and dynamic.

The outcomes

Those with advantage seek to perpetuate it. If a school or community presses for advantages regardless of the consequences for other schools then a rising tide of dissatisfaction is inevitable. Social cohesion is thrust apart.

In the USA bussing rich children to poor schools and vice versa was seen as the logical remedy to the problem of "under performing schools", perhaps Mr. Kemp would like to try this too. Half of Cranbrook School could go to Mt Druitt High and vice-versa. With exactly the same teachers Mt Druitt would experience a dramatic improvement of outcomes. Perhaps certain "Christian" schools should be expected to behave like Christ and minister to the needy as a prerequisite to Government funding. Surely all "private' schools should satisfy some public needs before dipping so readily into the public purse! Fear of failure is real for all of us but carrying a big stick is no solution. Playing statistics with little kids is bullyboy stuff.

The real questions the testers should be asking is how can we redress social imbalance? How can we most effectively educate our children and how do we provide a positive, innovative learning experience for particular children in their particular schools?

Steve Padgham is a parent of eleven-year-old David who hates tests. He is also a Mathematics teacher at Dickson College. The cartoon is by David.
josteve-padgham@msn.com.au

MORE ON TESTING

Another Extract from the AAMT Internet Mailing List Community March 2000

This piece was contributed to the online debate for and against paper and pencil testing earlier this year. It has been slightly edited for printing here as a stand-alone piece. The references to Clements and Ellerton refer to a paper against pencil and paper tests written by Nerida Ellerton and Ken Clements as a discussion starter for that debate.

It seems that there is little debate on the efficacies of pencil and paper testing. Most mathematics educators make the point that good tests, used well, are very useful tools to evaluate children's mathematical development and further needs.

Recent discussion in Australia has focused on the use of broad-based, externally set, pencil-and-paper mathematics tests. This is a sign that perceptions of teacher "accountability" (as if they had none previously!) have been privileged in the public arena over the learning of individual students and curriculum development in classrooms. It also signals that ways of reporting to systems are now seen to be more contentious than methods of communication with children and their parents.

This leap in focus is possible because of the nature of tests used by state and federal authorities. Testing every child (rather than using representative sampling or detailed evaluations carried out by teachers by teachers) requires the use of machine-marked, short-answer test items, each with only one correct answer. Clements and Ellerton get to the crux of the matter. Such tests provide teachers little opportunity for exploring how a student comes up with a right or wrong answer. Hence they give little indication of individual children's strengths to be built on, weakness that needs attention, or levels of understanding that can be used as a springboard for further learning. Thus most such tests do little to inform teaching or curriculum development, and do not provide enough detail for useful reporting of progress or for getting to the root of problems.

My Honours project in 1982, for example, took traditional test items and simplified the language of the questions, but not the mathematics involved. It showed that many of the children judged to be poor at mathematics were not. They were simply poor readers. The nature of current broad-based pencil and paper tests, as well as the way they are administered and marked, do not allow such analysis to be undertaken. The form of reporting tells such children (via their teachers and their parents who also do not know the root of the problem) that they are "bad at maths" in relation to their (average) peers.

A further vital concern raised by Clements and Ellerton is the impact that broad-based testing is likely to have on curriculum development. Expectations of both providers and consumers of education are influenced by the content and the form of national and state tests. I refer you to the recent comments of Geoff Masters, director of the Australian Council for Educational Research (ACER), who was reported to be concerned that the way national benchmark items are being used has the potential to "dumb down" expectations (The Australian, 25/2/2000). Perhaps he was not reported correctly, but nevertheless we should note that it is expected by national authorities as well as the item writers that only a small percentage of the children at the year level in question will fail "benchmark" tasks. If this is the only level and type of assessment that is valued and supported by the national government, then the future of Australian mathematics is at risk.

Note that such reduction of expectations does not happen only at system level, as some teachers I know have a bank of items from past LAPS tests (the Victorian tests given to Year 3 and 5 students) and now base some of their teaching around these. This is understandable. They want themselves, their classes, and their schools to be seen to be performing well. Other stakeholders, such as school administrators and parents, also benefit from this facade. What used to be merely one invaluable assessment tool for teachers has become the key attractor in an education sideshow.

Judy Mousley lectures in Mathematics Education at Deakin University.
email: judym@deakin.edu.au

STUDENTS, MATHEMATICS AND GRAPHIC CALCULATORS INTO THE NEW MILLENNIUM.

Sydney, 30-31 March 2000

This conference was organised by AAMT thanks to sponsorship by CASIO, Hewlett Packard, Sharp, and Texas Instruments. From the ACT I attended as the CMA representative along with Glenda Crick as the Department's representative, Peter McIntyre (ADFA), and M'Liss Jones. Approximately 80 delegates in all from the other States and Territories gathered for an intense look at how graphic calculators are being used across Australia. During the two days we attended two session of Master Classes, one on learning mathematics and one on assessing mathematics, presented by experienced practitioners with the tool. We also listened to panel discussion on pedagogy and methodology, the future in technology and participated in lively discussion groups following each of these sessions. A communiqué summarizing the discussions will be published.

As the CMA representative I would like to present my own observations to the membership. The graphic calculator is a powerful, purpose-built tool for teaching and learning, which I do not believe teachers in the ACT have a great deal of experience with. Nationally this is creating some debate, which was reflected in the various session types as outlined above (particularly assessment and equity.). Graphics calculators are becoming integral parts of learning not only in Senior Secondary, but also throughout the high school years.

VIC: A published list of approved calculators is distributed (basically those without QWERTY key boards or Symbolic manipulation) and they are allowed/assumed in all assessment tasks, and are not required to be cleared. On the equity issue, virtually all schools indicated that they could provide access once it was a Board requirement. Several schools reported on how this was done, often using loan/hire schemes as well as class sets.

WA: Tertiary Entrance exams are based on the assumption that students have access to approved calculators (similar to Victoria) with which they are familiar and which are not required to be cleared. The WA government set aside $0.5 million for inservicing of teachers during the implementation stages.

SA: Draft rewrite in progress for implementation in 2002/3. "SSABSA recommends that for 2001 all students have access to a graphics calculator and/or computer for school based assessment".

From 2002 they will also be assumed in exams. The budget will provide inservice assistance for teachers in 2002 to 2004 to help with the implementation.

NT: will follow S.A.

TAS: There is no explicit reference at present, but a review is commencing with target implementation 2002 and the general expectation is that graphic calculators will be specified.

QLD: Graphic calculators are not mandated but how to make them accessible is a school-based decision.

NSW: Currently Year 10 School Certificate allows graphics calculators (not QWERTY etc). A review is imminent and they will probably be allowed/mandated in some form.

From this brief summary I can see us becoming isolated and left behind!!

With the exception of one dissenting voice at the conference, there were no doubts as to the benefits of the use of the graphics calculator in the classroom for enhancing teaching and learning. There were many suggestions and practical examples of how to overcome the equity of access problems. A major stumbling block seems to be the concept of mandating their use and only then inservicing teachers. In the ACT this will probably be a school-based decision. The CMA will be endeavouring to offer inservices in the future and should also be able to help individual schools if desired. The AAMT Conference in January will also be an opportunity for this to occur.

For more information please contact me or one of the other participants.

Margaret Rowlands
Lake Tuggeranong College
margaret.rowlands@laketuggeranongs.act.edu.au

USING GRAPHICS CALCULATORS FOR DERIVATIVES AND INTEGRALS

Introduction

Graphics calculators lend themselves well to demonstrating the visual aspects of calculus, for example graphs of functions, tangent lines, areas under curves, etc, as well as to calculating numerically (approximating) many of the quantities that arise.

They can be used at a number of levels:

- As a basic graph plotter: what does the graph of $y=e^x$ look like?
- To investigate 'what if' questions: what happens if you change the parameters a and b in the equation $y=a\ e^{bx}$?
- To investigate (numerically) many of the basic calculations in calculus, such as finding derivatives at a point, definite integrals, maxima and minima, etc.
- To illustrate graphically, perhaps by way of a program, some of the concepts of calculus: for example, showing that we can approximate the area under a graph by the areas of some rectangles. With sufficient ingenuity, almost anything can be done here, the only limitation being the small screen of the calculator.
- To automate, using the built-in operations or programs, some of the calculations that arise in calculus: numerical integration methods, finding zeros and

intersections of curves, solving differential equations numerically, and so on.

At a more mundane level, graphics calculators are fun and relatively easy to use. Students pick up the operations very quickly, usually much faster than teachers, and if you can't get them to use a graphics calculator, there are heaps of games too. The other good news is that there are lots of resources available, many free on the Web, as well as a rapidly increasing number of books on using graphics calculators in almost every aspect of mathematics and science.

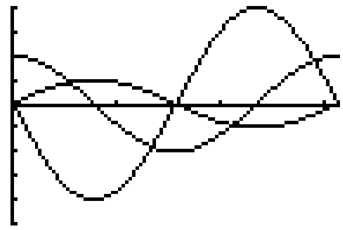
The experience at ADFA, and at most other schools and universities at which graphics calculators have been used for a while, is that they should not just be an add-on to a course; they should be integrated fully, including their use in tests and exams. This raises many issues, most of which are resolvable. A good discussion can be found in *'Graphics calculators in the mathematics curriculum: Integration or differentiation?'* by Jen Bradley, Barry Kissane and Marian Kemp about their experiences in WA, available at *wwwstaff.murdoch.edu.au/~kissane* under Publications.

Two Examples Involving Derivatives and Anti-derivatives[1]

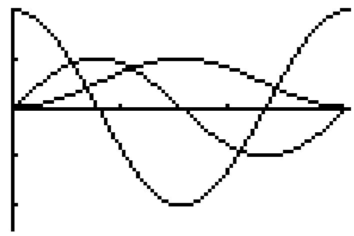
Here are two questions to challenge your students' understanding of the concepts of derivative and anti-derivative.

Question 1: The figure below plots a function, its derivative and its second derivative.

Which is which? Why?



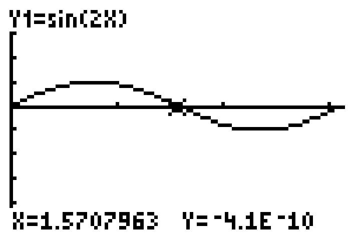
Question 2: The figure below plots a function, its derivative and an anti-derivative. Which is which? Why?



Graphing Functions

It's often interesting and instructive to look at the graphs of a function and its derivative and anti-derivative. Let's investigate how. The following instructions are for a TI-83 graphics calculator, but can be easily adapted to other brands.

Make sure that the calculator is in Radian MODE. Set $Y_1 = \sin(2X)$ and set a WINDOW $0 < X < \pi$, $-5 < Y < 4$, Xscl = Yscl = 1. Press TRACE to see a graph of the function.

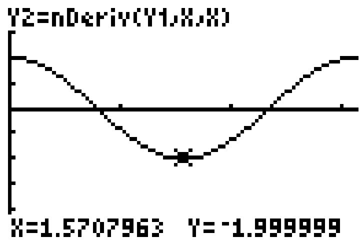


Graphing derivatives

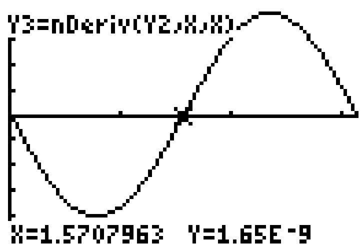
To graph the first derivative, set Y_2 = nDeriv(Y_1, X, X). *nDeriv*, in the MATH menu, uses the symmetric difference quotient to approximate derivatives. Y_1 is the function we are differentiating, the first

[1] Based on: John Maloney, *How do you graph derivatives and anti-derivatives?*, Eightysomething! 7(1), p5, 1997. Available at *www.ti.com/calc/docs/80xthing.htm*.

X tells the calculator we wish to differentiate with respect to variable X and the second X is the value at which we wish to calculate the derivative; this value is set by the grapher as it plots successive points on the graph.

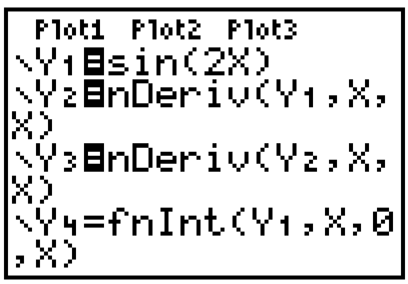


To graph the second derivative of Y_1, set Y_3 = nDeriv(Y_2, X, X), the first derivative of the first derivative.



The TI-83 will not allow you to calculate the third derivative in this manner; and it would not be very accurate anyway. A short program gets around this limitation.

With Y_1-Y_3 turned on, you should obtain the graph of the first question above.

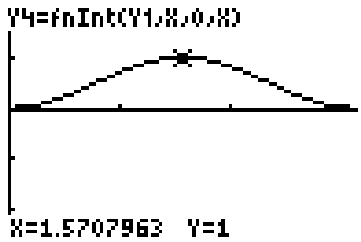


With the calculator set up in this manner, Y_2 and Y_3 give the first and second derivatives of whatever function is in Y_1.

Graphing anti-derivatives

To plot an anti-derivative of Y_1, set Y_4 = fnInt(Y_1,X,0,X). *fnInt*, in the MATH menu, is the built-in numerical integrator. Y_1 is the function we are integrating, X is the variable we are integrating with respect to, and 0 and X are the integration limits.[2]

Turn off Y_1-Y_3 (place the cursor over the = sign and press ENTER), set Ymin = -2.5, Ymax = 2 in WINDOW and press TRACE. Setting Xres = 3 (plotting every third point) speeds up this graph, without significant loss of resolution. Press ON to stop graphing.



The lower integration limit 0 is arbitrary. Change it to other numbers to show students that a function has (infinitely) many (related) anti-derivatives.

With Y_1, Y_2 and Y_4 turned on, you should obtain the graph of the second question above.

How Does ADFA Fit in to the Picture?

At University College ADFA, we have been using TI calculators (TI-83s at present) in our first-year courses since 1993 and have come to appreciate their worth in learning mathematics.

We are happy to help any school (free of charge) with:

2 This is poor notation, as we are using X for the dummy integration variable and for the independent variable (integration limit). Better is fnInt(sin(2T),T,0,X), giving the correct notation $\int_0^X \sin(2T)dT$, but this does not allow us to use Y1 as the function to integrate.

- assistance and backup in using any brand of graphics calculator;
- PD courses at your school on general or specific applications of graphics calculators;
- copies or loan of any resource materials we have;
- calculator programs for any activity;
- development of course material, in association with teachers.

Peter McIntyre
ADFA
Phone 6268 8896
p.mcintyre@adfa.edu.au

STATISTICIAN JOHN W. TUKEY DIES

July 26, 2000

PRINCETON, N.J. -- John Wilder Tukey, an emeritus Princeton professor considered to be one of the most important contributors to modern statistics, died Wednesday. He was 85.

Tukey developed many important tools of modern statistics and introduced concepts that were central to the creation of today's telecommunications technologies. In addition to his formidable research achievements, Tukey was known for his penchant for coining terms that reflected new ideas and techniques in the sciences and is credited with introducing the computer science terms "bit" (short for binary digit) and "software".

Tukey, Princeton's Donner Professor of Science Emeritus, actively applied his mathematical insights to real-world problems in engineering and social sciences, serving as staff researcher and associate executive director for research at Bell Labs, now the research and development arm of Lucent Technologies. For decades, he was an active consultant to such companies as Educational Testing Service and Merck & Co., and contributed to such areas as military operations in World War II, U.S. census-taking strategies and projecting the election-day results of presidential contests for national television.

"He probably made more original contributions to statistics than anyone else since World War II," said Frederick Mosteller, retired professor of mathematical statistics at Harvard University.

"I believe that the whole country -- scientifically, industrially, financially -- is better off because of him and bears evidence of his influence," said retired Princeton Professor John A. Wheeler, who is a major figure in the history of physics and the development of the atomic bomb.

"He had a penetrating understanding of so many areas in the field of statistics and was happy to share those insights with anyone who engaged him in a discussion," said David Hoaglin, a statistician at the social research firm Abt Associates who co-authored books and papers with Tukey. "It's hard to find an area that he did not work in or have a significant impact on."

Among Tukey's most far-reaching contributions was his development of techniques for "robust analysis," an approach to statistics that guards against wrong answers in situations where a randomly chosen sample of data happens to poorly represent the rest of the data set. Tukey also pioneered approaches to exploratory data analysis, developing graphing and plotting methods that are fixtures of introductory statistics texts, and authored many publications on time series analysis and other aspects of digital signal processing that have become central to modern engineering and science.

In 1965, with James Cooley, he introduced an analytical tool known as fast Fourier transform, which remains a ubiquitous technique for understanding waveforms in fields from astrophysics to electrical engineering.

In addition to his research achievements, Tukey was known for his passions for folk dancing and collecting murder mystery and science fiction books.

"John was a very lively presence on campus," said Princeton Professor of Mathematics Robert Gunning, former chairman of the mathematics department and dean of the faculty.

In one commonly told anecdote, Tukey put his extraordinary calculating abilities to work as chairman of the Faculty Committee on Schedule, working out the seemingly intractable complexities of arranging times for classes and exams.

"He would lie flat on his back on a table and people would list the scheduling difficulties and he would reel off solutions," Gunning said. "He did it quickly and quietly in his head."

Tukey also was instrumental in creating a citation index for statistical literature and was known for carrying publication lists with him and working out the complexities of cross-references in his spare time.

"He did an amazing number of things," Gunning said. "And he was a good and energetic teacher."

"If you have money in the bank you always have a sense of assurance," said Wheeler. "John Tukey was a special kind of money in the bank because you could take up a difficult question with him and get a new point of view and sound advice. The country will be poorer for his loss."

Tukey was born in New Bedford, MA on June 16, 1915. He earned bachelor's and master's degrees in chemistry from Brown University in 1936 and 1937 before coming to Princeton for graduate work in mathematics. He earned his Ph.D. in just two years. After spending wartime years in the government's Fire Control Research Office in Princeton, Tukey rose to the rank of full professor by 1950 at age 35.

Building on a foundation laid by statistician Samuel S. Wilks, Tukey helped found a department of statistics, which split from the mathematics department in 1966, and chaired the department until 1970. The department later became today's Committee for Statistical Studies.

Among many awards and honors, Tukey received the National Medal of Science in 1973 and an honorary doctorate from Princeton in 1998, and was a member of the National Academy of Sciences and the Royal Society of England.

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USING METAPHOR ANALYSIS TO EXPLORE ADULTS' IMAGES OF MATHEMATICS

This paper discusses the possible use of metaphor analysis in exploring adults' images of mathematics. Based on reviews of past literature and the author's own experience of using this kind of analysis in one of the image study, the potential advantages and problems encountered during the study are discussed.

What is a metaphor?

Metaphors are pervasive in our daily life. We use metaphors to conceptualise, to represent and to communicate many of our thoughts and actions (Lakoff and Johnson, 1980). According to Lakoff and Johnson (1980), a metaphor is a mental construction that helps us to structure our experience and to develop our imagination and reasoning. Moreover, according to Johnson (1987) metaphors are constructed through an 'embodied schema' or an 'image schema'. An embodied schema is defined as "structures of an activity by which we organize our experience in ways that we can comprehend. They are primary means by which we construct or constitute order and not mere passive receptacles into which experience is poured" (Johnson, 1987, pp.29-30). This means we construct metaphors to link our bodily experience of something to our more abstract thinking, and to "give shape, structure, and meaning to our imagination" (Sfard, 1994, p.47). This suggests that in fact, the whole conceptual system of how we think and act may be fundamentally metaphorical in nature. In brief, metaphors are something that constructed by our minds that help us to present something in terms of something else. For example, Sfard (1996) proposes that two metaphors for mathematics learning as acquisition and participation metaphors.

Reviews on the use of metaphors

Reviews on past related literature have shown that the use of metaphor in scientific thinking (e.g. Ortony, 1979, Martins & Ogborn, 1997) and as an analysis tool for mathematical thinking (Pimm, 1981, 1987) has been recognised for some time. Recently, there is an increasing number of studies using metaphor to explore teachers' thinking (e.g. Cooney, et al., 1985; Munby, 1986; Clandinin's, 1986; Bullough, 1991) and mathematicians' experience of understanding mathematics (Sfard, 1994). For example, Munby (1986) studied some teachers' use of metaphors in their descriptions of their work, and he concludes that "given the powerful link between metaphor and construction of reality" (p.206), the use of metaphors prove to be a promising alternative in exploring teachers' thinking.

Likewise, Clandinin's (1986) study used metaphors such as 'classroom as home' and 'language as the key' to conceptualise teachers' personal practical knowledge and their classroom practice. In another study, metaphors such as 'teaching as parenting'; 'teacher as butterfly' and 'teacher as chameleon' are used to explore preservice teachers' personal teaching experience (see Bullough, 1991).

In fact, the use of metaphors to describe images of mathematics or learning mathematics is also not rare. Kelly and Oldham (1992) studied the images of mathematics and mathematics education of a group of primary school teachers and student teachers. They found that the metaphor best fitted the overall picture in their sample's image of mathematics and mathematics education was 'a racecourse round which students and teachers had to gallop, generally jumping hurdles (problem solving) on the way'. Likewise, Buerk (1982) noted the metaphors presented by a group of able women who avoid mathematics. These included the following:

Mathematics does make me think of a stainless steel wall-hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, put your nose against it, it doesn't give anything back, you can't put a dent in it, it doesn't take your shape, it doesn't have any smell, all it does is make your nose cold. I like the shine of it - it does look smart, intelligent in any icy way. But I resent its cold impenetrability, its supercilious glare. (p.19)

In addition, Sfard (1996) proposes that "mathematics education research seems to be caught in between two metaphors," which he calls them as '*Acquisition Metaphor* and *Participation Metaphor*" (p. 399). According to Sfard (1996), learning-as Acquisition Metaphor has been deeply entrenched in our thinking about learning mathematics. For example, we uses titles such as "Acquisition of mathematical concepts and processes, Building up mathematics,…" (p.400) for many major publications in mathematics education. It is also commonly quoted that "the teacher may help the student to attain her goal by *delivering, conveying, facilitating, mediating* etc." (p.400). All these expressions suggest that mathematics is viewed as an accumulated commodity and thus learning mathematics is equated with acquisition of this commodity. However, recently there has been a shift in the vision of learning mathematics where the metaphor of 'learning-as-participation' has become more apparent. Unlike the acquisition metaphor, learning as participation metaphor stresses the importance of learner as becoming a participant, rather than as acquisition of something. Nevertheless, Sfard (1996) concludes that there is no clear-cut preference from one metaphor to another, but more important, 'the acquisition and participation metaphor, when combined together, run a good chance of gratifying all our needs without perpetuating the drawbacks of each one of them" (p.409).

An example of a study which uses metaphor analysis

In a study to explore the images of mathematics of a sample of adults of the UK public (see Lim, 1999), I chose to use metaphor analysis as one of my methods to analyse the data. This is in view with the plausibility of associating metaphors with image (as discussed in the review above). Defining image as a mental construct, Rogers (1992) argued that throughout history, philosophers and mathematicians have been involved in the ontological questions about the status, 'reality' and existence of mental images. Whichever philosophical standpoint we take, we have to admit two fundamental aspects of this debate: first, we are aware of the power of the human mind to construct mental images; and secondly, our abilities to manipulate these images and use them to inspire creative thought, and many different forms of communication" (p.49).

Therefore, I would expect that some people might use metaphors to describe their images. This is also because it is this human desire to make stories about the world in our attempts to come to terms with the physical and metaphysical phenomena we encounter daily has led to a vast fund of metaphor; of manipulating our images to enable us to come to terms in some way with the world we live in. (Rogers, 1992, p.50)

Thus, in the study, I argued that it is plausible to conceptualised image as metaphor and to expect some of the respondents to express their images of mathematics in terms of metaphors. Here I describe briefly how I carried the study and the problems encountered as a result of this kind of analysis.

The study used a short questionnaire and a followed-up interview to collect data. The questionnaire consisted of two open-ended questions and 9 structured questions. The open-ended questions asked for participants' images of mathematics and learning mathematics in the form of descriptions, and metaphors or analogies.

Initially I hesitated as to whether I should give an example for each of these open-ended questions, as I understand that any example given might bias the responses obtained. However, during piloting, I was asked for an example of an image by a number of respondents. Thus, I decided to give an example of metaphor to act as a trigger as well as to clarify the question. I acknowledge that whatever type of example that I give (such as positive or negative image) might influence the type of answer that I obtain. So I chose to give an example of an image in the form of metaphor, and which was neutral with regard to positive or negative images of mathematics.

What did I get?

As expected, the two open-ended questions elicited a wide range and variety of responses. In what follows, each individual respondent has been identified by a unique code Rabc, where a,b,c denote digits. Some responses expressed views or proposition such as mathematics is "interesting but difficult " (R329) or "a lot of things that I will not ever use" (R059). Others were given in metaphor forms such as mathematics is "like playing with my children, never tiresome" (R526) and learning mathematics is "learning to walk, we've all got to " (R009). In the study, 27% of the respondents expressed their images of mathematics in the forms of metaphors, while 66% of them gave their images of learning mathematics in metaphoric terms. It was exciting to find the variety and diversity of these metaphors, besides the commonness that they shared. However, the rich variety and diversity of metaphors have also given rise to problem of multiple interpretations.

First of all, I sought to find a common theme that emerged from the data. I found although some responses were given in metaphoric forms, they share some commonness and can be classify into some main categories. Three common categories of metaphors emerged from the data. These were:

a) Mathematics as a journey

b) Mathematics as a skill

c) **Mathematics as a game or puzzle**

The analysis of the study shows that the metaphor of mathematics-as-journey seems to be most commonly expressed by the respondents. Some examples are:

Mathematics is a

> challenging journey - rewarded by arrival at your destination (R255).

Learning mathematics is like

> an easy stroll on a windy day (R034)
>
> running uphill - difficult but you get there (R376)

Implicitly, the journey metaphor highlighted the close relationship between images of mathematics and images of learning mathematics. For many, viewing mathematics as a challenging journey elicits the experiences or the process of learning mathematics. For some people, the experience of learning mathematics might be like a struggle in a journey such as, "walking through mud" (R155) or "an uphill struggle" (R417). These metaphors indirectly indicate the difficulty and frustration that were experienced by these respondents, especially those reporting a dislike in learning mathematics in school. Some of them felt that learning mathematics is like "being stuck in a bus queue" (R268).

In contrast, particularly those who reported a liking of mathematics viewed these journeys as explorations or discoveries. For them, learning mathematics is like

> exploring - there is always something new to know (R331) or

> being an explorer-finding new paths and worlds (R364)

For these people, mathematics is a journey to discover new things, new knowledge and new insights. These results suggest that it was the joy of discovering new understanding in mathematics that attracted them to get interested in mathematics. Even though many of them also found learning mathematics a difficult journey like,

> a journey through a dark tunnel with a light at the end (R139) or

> walking through sand - hard work but put in effort, you'll get there (R136)

Therefore, there is this sense of achievement and satisfaction that encourage these people to work hard and to strive for solution. Implicitly these metaphors indicate that there is a definite solution for each mathematics problem. Learning mathematics is "a journey through a dark tunnel with a light at the end" (R139) and there is a destination for you "to get there" (R133, text-unit 13).

It is interesting to read that some undergraduate students and tutors in Allen and Shiu's (1997) study also gave similar metaphors that reflect mathematics as a journey. Allen and Shiu (1997) categorised these responses under one of their four categories: "struggle leading to success". Two very similar responses from the tutors are: learning mathematics is like

> climbing a hill: - hard work where you follow the path you're on - and then the joy and satisfaction of being at the top (T3)

> climbing a hill. The higher you get the clearer the view of surrounding countryside - as you can see more the links and layout and connections become more obvious. (T18). (p.10)

In short, 'mathematics as a journey' metaphor indicates that mathematics learning is a difficult process that needs a lot of effort and time. However, there are two possible extreme outcomes: either you reach the destination (obtain the solution) and feel happy and satisfied, or the opposite, fail to solve the problem and feel disappointed and frustrated.

Mathematics as a skill was the next commonly expressed metaphor. Closely linked to a utilitarian view of mathematics, some images of mathematics portray mathematics as an important and necessary skill for daily life and work. Mathematics is an essential basic skill for society (R092).

Similarly, learning mathematics is like:

> learning to walk, we've all got to (R009).

Once again, the skill metaphors reflect the view that mathematics is a skill that is not always easy to learn, just like

> learning a musical instrument, some are easier and others are extremely hard (R542).

Nevertheless, at least nine respondents were attracted to learning mathematics because to them, learning mathematics is acquiring a skill, like "riding a bike - once learnt never forgotten" (R123).

There were some respondents who viewed mathematics as a set of skills that is hierarchical, like " brick laying - each brick is the foundation for the next block" (R081).

Others believed that mathematics learning is a skill that needs

> memorisation such as "learning law: rules and cases to remember in total" (R485) or

> needing a lot of practice: "learning to ride a bike - takes plenty of practice" (R520).

Likewise, the skill metaphor for mathematics suggests that learning it could be a skill that develops more easily for some people, like

> playing the stock exchange - once you get the hang of it, it's ok (R469),

or gets more difficult for others, just like

> riding a bike, simple enough until you come to a mountain (R066).

In summary, 'mathematics as a skill' metaphor suggests that mathematics is viewed in terms of its utilitarian value, while learning mathematics viewed as a skill is seem to be hierarchical, needing memorisation and lots of practice; difficult to be mastered by some but easy for others.

The third category of metaphor was viewing mathematics as a game or puzzle. This seems to be closely related to the problem solving views. Examples are mathematics is "a brain teaser - a puzzle to be solved" (R388) and learning mathematics becomes

> finding your way through the maze (R174);

> playing chess - absorbing and challenging (R220).

Viewing mathematics as a game or a puzzle to be solved reflects the fact that mathematics learning is fun and challenging for some people. Mathematics is

> fun when everything works out but remains a challenge (R470)

or learning mathematics is like playing

> a jigsaw puzzle - slow but relaxing- it makes your mind work (R389)

What are the problems?

The above quoted three commonly expressed metaphors – as a journey, as a skill and as a game or puzzle – were rather clear-cut. However, there were also a lot of responses that were opened up to various possible interpretations. For example, learning mathematics is like "playing with my children never tiresome" (R526) or "like going to sleep" (R003). Should we interpret these metaphors as daily life experiences that are enjoyable and not enjoyable respectively? Can we argue that these words ‘never tiresome’ indicate ‘enjoyable’ while ‘going to sleep" indicate ‘boring and thus not enjoyable’? However, as one of my colleagues pointed out that to her, ‘going to sleep’ is an enjoyable experience especially when you are feeling tired. This raises problems concerning the validity of our interpretation and data analysis.

In brief, I face at least three problems while I was trying to analyse the responses. These problems include:

a) **it may opens to too many possible interpretations;**

b) **some responses are too ambiguous and abstract to be interpreted; and**

c) **one metaphor may be interpreted differently by different researchers.**

For example, in the study, one response to the question on the image of mathematics was given as 'maths is a snail shell in the garden' (R117). It was too ambiguous. "A snail shell" may be taken as unused rubbish in the garden and thus mathematics is equated to useless in daily life? One could also argue that "a snail shell" may be interpreted as something that is commonly found in the garden and therefore mathematics is something essential in daily life? It opens up to too many possible interpretations and it was not possible to get further confirmation because the respondent did not agree to be interviewed in the second stage of this study. As it is too difficult to interpret what exactly the metaphor is implying. Therefore, for this type of response, the best way is to disregard it.

On the other hand, another response given was "'peaches and cream - solid basic sweet effect ' (R544) which is equally ambiguous as the previous one. However, it was possible to reconfirm and clarify its meaning with the respondent because he agreed to take up the follow-up interview in the second stage of the study. At first, I coded it in the category of 'beauty of mathematics'. During the interview, we have the following conversation:

> I: You mentioned that mathematics is 'peaches and cream - solid basic sweet effect', could you please explain what you meant by this metaphor?
>
> R544: Yes, It seems to me that people that don't like it, see it as a bit frightening and horrific, they got a hang up because in the past, they always failed to come out with a right answer. If you got a fairly open-ended mind with regard to things like maths and science, then you won't ended seeing it as necessary having to come out with a concrete conclusion. So, that aspect of it I don't find any of it frightening. In the same way that if you use the metaphor like peaches and cream as oppose to hmm... something like fish and chips, you actually like fish and chips because it is better taste that you get with vinegar and stuff like that. You see the difference? That is how I will make the comparison. I suspect that people who don't like it, hmm, would like much vinegar on their fish and chip. Ha! Ha! [laughter]

Therefore, after the interview, I have recorded his response to the category of 'enjoyable'. The possibility of misinterpretation like the above shows the importance of reconfirmation with the respondent whenever it is possible.

Conclusion

The rich variety of images in the form of metaphors, and other verbal representations have illustrated the possible use of metaphors analysis, as a mean to gain a better insight and understanding into people's conceptions, views, feelings, and their experiences related to mathematics education. However, the wide range in variety also opens up the problem of multiple interpretations. Therefore, effort such as reconfirmation with the respondents may be a way to overcome these problems.

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Chap Sam Lim
School of Educational Studies
University of Science Malaysia
cslim@usm.my

THE POWER OF PARTITIONS

Writing a whole number as the sum of smaller numbers springs a mathematical surprise

Just a year before his death in 1920 at the age of 32, mathematician Srinivasa Ramanujan came upon a remarkable pattern in a special list of whole numbers.

The list represented counts of how many ways a given whole number can be expressed as a sum of positive integers. For example, 4 can be written as 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. Including 4 itself but excluding different arrangements of the same integers (2 + 1 + 1 is considered the same as 1 + 2 + 1), there are five distinct possibilities, or so-called partitions, of the number 4. Similarly, the integer 5 has seven partitions.

The list that Ramanujan perused gave for each of the first 200 integers, the number of their partitions, which range from 1 to 3,972,999,029,388.

Ramanujan noticed that, starting with 4, the number of partitions for every fifth integer is a multiple of 5. For instance, the number of partitions for 9 is 30 and for 14 is 135.

He discovered several more such patterns. Starting with 5, the number of partitions for every seventh integer is a multiple of 7, and starting with 6, the number of partitions for every 11th integer is a multiple of 11. Moreover, similar relationships occur where the interval between the chosen integers is a power of 5, 7, or 11 or a product of these powers. Ramanujan went on to prove rigorously that these patterns hold not only for the 200 partition numbers in his table but also for all higher numbers.

It was a curious discovery. Nothing in the definition of partitions hinted that such relationships, called congruences, should exist or that the prime numbers 5, 7, and 11 should play a special role.

After many decades of nearly fruitless searching that yielded just one or two apparently isolated examples of large numbers that fit the pattern, mathematicians came to believe that no other congruences exist. Those found by Ramanujan and the later mathematicians were thought to be little more than numerical flukes.

Indeed, "it was really believed that there would probably never be any new major discoveries regarding partition congruences," says George E. Andrews of Pennsylvania State University in State College.

In a startling turnabout, mathematician Ken Ono has now proved that infinitely many such relationships occur. "This was really quite unexpected," says number theorist Ono, who holds positions at both Penn State and the University of Wisconsin-Madison. He described his results in the January *Annals of Mathematics*.

"Ono's work is really spectacular," comments Bruce C. Berndt of the University of Illinois at Urbana-Champaign. "This certainly must rank as the most important work on partition congruences since the epic work of Ramanujan."

"It's an extraordinary discovery," agrees Andrew M. Granville of the University of Georgia in Athens.

Moreover, Ono's partition research has intriguing, unexpected links to complex mathematical ideas and methods that earlier led to proofs of Fermat's last theorem (SN: 11/5/94, p. 295) and the Taniyama-Shimura conjecture. It has opened up new avenues for studying some of the most important, but difficult, questions in number theory, Granville says.

Born into poverty, Ramanujan grew up in southern India, and although he had little formal training in mathematics, he became hooked on mathematics. He spent the years between 1903 and 1913 cramming notebooks with page after page of mathematical formulas and relationships that he had uncovered).

Ramanujan's life as a professional mathematician began in 1914 when he accepted an invitation from the prominent British mathematician G.H. Hardy to come to Cambridge University. He spent 5 years in England, publishing many papers and achieving international recognition for his mathematical research.

Though his work was cut short by a mysterious illness that brought him back to India for the final year of his life, Ramanujan's work has remained a subject of considerable interest. For the past 2 decades, Berndt has been going through Ramanujan's scribbled notes, systematically deciphering, writing out, and proving each of the numerous theorems Ramanujan had formulated.

Two years ago, Berndt was examining an unpublished, handwritten Ramanujan manuscript on partitions. "It contained claims that had never been proved," Berndt remarks. So he recruited Ono, an expert on partitions from a modern perspective, to help him fill in the details and provide the necessary proofs.

"I had never read any of Ramanujan's notebooks, though I was familiar with a lot of what he had done through the writings of more modern mathematicians," Ono says. "I didn't suspect that I would learn anything from studying Ramanujan's notes."

In fact, the manuscript didn't contain anything startling or new. However, Ramanujan had written out one mathematical expression, or identity, in a manner that to Ono seemed particularly awkward.

"The first time I saw it, I wrote to Berndt saying this can't be right," Ono recalls. Nonetheless, after Ono translated the expression into modern mathematical language, it made perfect sense.

Partitions of 5

The integer 5 has seven partitions:

5

4 + 1

3 + 2

3 + 1 + 1

2 + 2 + 1

2 + 1 + 1 + 1

1 + 1 + 1 + 1 + 1

Yet the apparent awkwardness of Ramanujan's original formulation bothered Ono. It got him to think deeply about the many different ways in which mathematicians can express identities. In the course of that rethinking, he obtained crucial insights that led him to tackle the question of partition congruences—something that he would not otherwise have attempted.

"I learned a valuable lesson," Ono remarks. "It sometimes really pays to read the original."

In a remarkable tour de force, Ono proved that partition congruences can be found not only for the prime numbers 5, 7, and 11 but also for all larger primes. To do so, he had to exploit a previously unsuspected link between partition numbers and complex mathematical objects called modular forms.

"Modular forms lie at the heart of modern number theory," says Scott Ahlgren of Colgate University in Hamilton, N.Y.

They are special types of mathematical relationships that involve so-called complex numbers, which have a real and an imaginary component. In effect, the relationships represent particular types of repeating patterns, roughly analogous to the way a trigonometric function such as sine represents a repeating pattern for ordinary numbers.

Mathematicians have identified many different types of modular forms. The connection between certain modular forms and elliptic curves helped Andrew Wiles of Princeton University prove Fermat's last theorem in 1993.

Applying modular-form theory, as developed by other mathematicians, Ono uncovered a connection between partition numbers and specific modular forms. He used that relationship to establish, in effect, the existence of congruences involving prime-number divisors among the partition numbers.

Unlike many previous advances in partition theory, Ono's research involved no computation and relied heavily on theory. "What I find particularly appealing about this approach is that it uses the most powerful tools in modern number theory to attack a classical problem," Ahlgren says.

Interestingly, although Ono proved the existence of these congruences, he provided only one explicit example. In this new congruence, the starting number is 111,247, with each successive step to the next integer going up by 59^4 x 13. The corresponding partitions are then multiples of the prime number 13.

Ono, however, didn't have an effective recipe, or algorithm, for generating examples. But Rhiannon L. Weaver, an undergraduate at Penn State, found a simple criterion for identifying the start of a progression. She then developed an algorithm, working with primes from 13 to 31, to obtain more than 70,000 new congruences.

Ramanujan Congruences

The idea that starting with 4, the number of partitions for every fifth integer is a multiple of 5 is due to Ramanujan. This particular result can be expressed compactly in the following mathematical form:

for N greater than or equal to 0, $p(5N+4) \equiv 0 \pmod{5}$.

Here p(n), called the partition function, is the number of partitions of an integer n; ≡ indicates congruence; and 0 (mod 5) means that the remainder after division by 5 must be 0.

Ramanujan discovered other such patterns. Recently undergraduate student at Pennsylvania State University Rhiannon L. Weaver discovered many more, for example:

$p(11864749\ N + 56062) \equiv 0 \pmod{13}$

and $p(14375\ N + 3474) \equiv 0 \pmod{23}$

"This wasn't a trivial exercise," Ono says. "It was a great piece of work." Applying Weaver's method, a researcher can now readily write a computer program to find thousands of additional examples, he explains.

The newfound congruences also show why mathematicians had failed to come up with many additional examples by trial and error. "The numbers involved are very big," Granville remarks. Moreover, "even now that we know where to look, I don't think we would have spotted them from raw computation."

In another recent development, Ahlgren has extended Ono's results to establish the existence of congruences that work with multiples of composite numbers, which consist of primes multiplied together, as long as the numbers aren't divisible by 2 or 3.

Ahlgren reported his results in May at the Millennial Conference on Number Theory held at Illinois.

"It is now apparent that Ramanujan-type congruences are plentiful," Ono says. "However, it is typical that such congruences are monstrous."

Ono's results have already sparked a considerable amount of research activity. "It's amazing how much more we know now than we did just last year," Ahlgren says.

Despite these advances, however, mathematicians still don't know whether there are congruences that involve multiples of 2 or 3. Ono's methods don't work for these particular cases, and researchers must develop new tools to hunt for such relationships.

At the same time, Ono has given partition numbers an exciting, new role in mathematics. "Partitions are much more than just counts of how to add up numbers," Ono says. "They are a vehicle for testing some of the most important conjectures about [mathematical] objects that we can barely get a handle on otherwise."

Indeed, studying partitions could lead to new insights into the theory of modular forms and illuminate its connections with important, unsolved questions in number theory, such as the so-called Swinnerton-Dyer conjecture. The Clay Mathematics Institute in Cambridge, Mass., recently listed this problem as one of the top seven questions in mathematics and offered a $1 million prize for its solution. The conjecture concerns a criterion for deciding whether certain equations have whole-number solutions.

Integer	Number of Partitions
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
10	42
11	56
12	77
13	101
14	135
15	176
16	231
17	297
18	385
19	490
20	627
21	792
22	1,002
23	1,255
24	1,575
25	1,958
26	2,436
27	3,010
28	3,718
29	4,565
30	5,604
31	6,842
32	8,349
33	10,143
34	12,310
35	14,883
36	17,977
37	21,637
38	26,015
39	31,185
40	37,338
41	44,583
42	53,174
43	63,261
44	75,175
45	89,134

The number of partitions for each integer from 1 to 45. The box colored green represents a partition number that's a multiple of both 5 and 7, and the orange box indicates a multiple of 5 and 11. Yellow boxes indicate those numbers that are multiples of only 5; blue represents multiples of only 7; and red indicates multiples of only 11.

The study of partitions has long been one of the mainstays of number theory and rivals the study of primes for intrinsic mathematical appeal, Andrews says.

"Primes form the basis for multiplication, and the study of partitions is grounded in the addition of integers," he notes. "What is intriguing is the fact that such an elementary idea can have theorems as subtle as those of Ono."

Ivars Peterson
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WILL THE REAL CONTINUOUS FUNCTION PLEASE STAND UP?

Devlin's Angle

What exactly is a continuous function? Here is a typical explanation taken from a university textbook (G. F. Simmons, *Calculus with Analytic Geometry,* McGraw-Hill, 1985):

In everyday speech, a 'continuous' process is one that proceeds without gaps of interruptions or sudden changes. Roughly speaking, a function $y = f(x)$ is continuous if it displays similar behavior, that is, if a small change in x produces a small change in the corresponding value $f(x)$.

As the author observes, this description is "rather loose and intuitive, and intended more to explain than to define." He goes on to provide a "more rigorous, formal definition," which I summarize as:

A function f is continuous at a number a if the following three conditions are satisfied:

1. f is defined on an open interval containing a.
2. tends to a limit as x tends to a.
3. That limit is equal to $f(a)$.

To make this precise, we need to define the notion of a limit:

If a function $f(x)$ is defined on an open interval containing a, except possibly at a itself, we say that f tends to a limit L as x tends to a, where L is a real number, if, for any epsilon > 0, there is a delta > 0 such that:

if 0 < $|x - a|$ < delta, then $|f(x) - L|$ < epsilon

With limits defined in this way, the resulting definition of a continuous function is known as the Cauchy-Weierstrass definition, after the two nineteenth century mathematicians who developed it. The definition forms the bedrock of modern real analysis and any standard "rigorous" treatment of calculus. As a result, it is the gateway through which all students must pass in order to enter those domains. But how many of us manage to pass through that gateway without considerable effort? Certainly, I did not, and neither has any of my students in twenty-five years of university mathematics teaching. Why is there so much difficulty in understanding this definition? Admittedly the logical structure of the definition is somewhat intricate. But it's not *that* complicated. Most of us can handle a complicated definition provided we understand what that definition is trying to say. Thus, it seems likely that something else is going on to cause so much difficulty, something to do with what the definition *means.* But what, exactly?

Let's start with the intuitive idea of continuity that we started out with, the idea of a function that has no gaps, interruptions, or sudden changes. This is essentially the conception Newton and Leibniz worked with. So too did Euler, who wrote of "a curve described by freely leading the hand." Notice that this conception of continuity is fundamentally dynamic. Either we think of the curve as being *drawn* in a continuous (sic) fashion, or else we view the curve as already drawn and imagine what it is like to *travel* along it. This means that our mental conception has the following features:

1. The continuous function is formed by motion, which takes place over time.
2. The function has directionality.
3. The continuity arises from the motion.
4. The motion results in a static line with no gaps or jumps.
5. The static line has no directionality.

Aspects of this dynamic view are still present when we start to develop a more formal definition: we speak about the values *f(x) approaching* the value *f(a)* as *x approaches a*. The mental picture here is one of preserving closeness near a point.

Notice that the formal definition of a limit implicitly assumes that the real line is continuous (i.e., gapless, or a continuum). For, if it were not, then talk about *x* approaching *a* would not capture the conception we need. In this conception, a line or a continuum is a fundamental object in its own right. Points are simply locations on lines.

When we formulate the final Cauchy-Weierstrass definition, however, by making precise the notion of a limit, we abandon the dynamic view, based on the idea of a gapless real continuum, and replace it by an entirely static conception that speaks about the existence of real numbers having certain properties. The conception of a line that underlies this definition is that a line *is* a set of points. The points are now the fundamental objects, not the line. This, of course, is a highly abstract conception of a line that was only introduced in the late nineteenth century, and then only in response to difficulties encountered dealing with some pathological examples of functions.

When you think about it, that's quite a major shift in conceptual model, from the highly natural and intuitive idea of motion (in time) along a continuum to a contrived statement about the existence of numbers, based on the highly artificial view of a line as being a set of points. When we (i.e., mathematics instructors) introduce our students to the "formal" definition of continuity, we are not, as we claim, making a loose, intuitive notion more formal and rigorous. Rather, we are *changing the conception of continuity* in almost every respect. No wonder our students don't see how the formal definition captures their intuitions. It doesn't. It attempts to replace their intuitive picture with something quite different.

Perhaps our students would have less trouble trying to understand the Cauchy-Weierstrass definition if we told them in advance that it was not a formalization of their intuitive conception -- that the mathematician's formal notion of a continuous function is in fact something quite different from the intuitive picture. Indeed, that might help. But if we are getting into the business of open disclosure, we had better go the whole way and point out that the new definition does not explicitly capture continuity at all. That famous -- indeed, infamous -- epsilon-delta statement that causes everyone so much trouble does not eliminate (all) the vagueness inherent in the intuitive notion of continuity. Indeed, it doesn't address *continuity* at all. Rather, it simply formalizes the notion of "correspondingly" in the relation "correspondingly close." In fact, the Cauchy-Weierstrass definition only manages to provide a definition of continuity of a function by assuming continuity of the real line!

It is perhaps worth mentioning, if only because some students may have come to terms with the idea that a line is a set of points, that in terms of that conception of a line -- which is not something that someone or something can move along -- the original, intuitive idea of continuity reduces simply to gaplessness. In short, however you approach it, the step from the intuitive notion of continuity to the formal, Cauchy-Weierstrass definition, involves a huge mental discontinuity.

This article is based on the paper *Embodied cognition as grounding for situatedness and context in mathematics education,* by R. Nunez, L. Edwards, and J. Matos, *Educational Studies in Mathematics* 39 (1999), pp.45-65.

Keith Devlin contributes Devlin's Angle to MAA Online at www.maa.org
He is a Senior Researcher at Stanford University. His latest book is *The Maths Gene: Why Everybody Has It But Most People Don't Use It*. This piece dates from May 2000.

QUOTABLE NOTES AND NOTABLE QUOTES

From Numbers to Geometry:
Two Views on the Poetry of Life

Love Poem

by Judith Wright

Counting in Sevens
Seven ones are seven
I can't remember that year
or what presents I was given.

Seven two's are fourteen.
That year I found my mind,
swore not to be what I had been.

Seven threes are twenty-one.
I was sailing my own sea,
first in love, the knots undone.

Seven fours are twenty-eight;
three false starts had come and gone.
My true love came, and not too late.

Seven fives are thirty-five.
In her cot my daughter lay,
real, miraculous, alive.

Seven sixes are forty-two.
I packed her sandwiches for school
I loved my love and time came true.

Seven sevens are forty nine.
Fruit loaded down my apple tree,
near fifty years of life were mine.

Seven eights are fifty-six.
My lips still cold from a last kiss,
my fire was ash and charcoal sticks.

Seven nines are sixty-three,
seven tens are seventy.
Who would that old woman be?
She will remember being me,
but what she is I cannot see.

Yet with every added seven,
some strange present I was given.

Geometry

by David Martin

To drink a glass of wine is good
But to drink two is better.
To love two women is like loving
Christ and the Devil together.

To make a song of love is sweet
But to make love is sweeter.
Who loves two women finds his song
Grow inward and turn bitter.

Who seeks to make the circle square
Shall see his reason vanish.
But who would square the triangle
Venus will blind and punish.

A Sure Route to Understanding

'You understand, because you succeed in making another understand.'

J. H. Fabre in *The Life of the Fly*

"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense and even mysterious. But once unlocked, they cast a brillian new light on some aspect of computing."

Francis Sullivan

Mathematics: Something From Nothing

"One cannot escape the feeling that these mathematical formulas have an independent existence of their own, and they are wiser than even their discoverers, that we get more out of them than was originally put into them."

Heinrich Hertz, the first to produce
and detect radio waves in the lab.

A Sure Bet

Perhaps Ladbrokes, the British bookmaker, should employ more mathematicians.

At the end of last month, the company offered a free £50 bet to anyone who deposited at least £50 in their betting account. So if you bet £50, win or lose, they gave you another £50 to bet. After you had made a second bet with this free £50 you could withdraw any money left in your account.

A friend of *Feedback* - let's call him Neil - realised that if he worked in a pair he could ensure them both a profit. He and his friend Stuart bet on tennis, which is an either/or - there can only be two possible results. By betting against each other - for example, if Henman played Sampras, Neil would bet on Henman and Stuart would bet on Sampras - then irrespective of the result one of them would win. The trick is to make two bets like these at £50 each, one of the original stake and one with the free £50.

So, for example, if Neil's bets both lost, Stuart would have made two winning bets and his account would contain £100 (giving them both their stakes back) plus profit, to be split between them, and vice versa. Making one winning bet of £50 each meant each final account would contain £50 - cancelling out the original stake - plus the profit from the winnings, as determined by the odds.

Neil and Stuart are now celebrating in style.

Opinion Feedback,
New Scientist,15 July 2000

Gristly Undecidability

"Undecidable propositions run through mathematics like threads of gristle that criss-cross a steak in such a dense way that they cannot be cut out without the entire steak's being destroyed."

Douglas Hofstadter

Napoleon Versus the Metric System

In exile on St. Helena Napoleon indulged in a tirade against the metric system, beginning with the folly of consulting mathematicians 'upon a question which was wholly within the province of the administration ... Nothing can be more contrary to the organisation of the mind, the memory and the imagination [than the metric system] ... The new system of weights and measures will be the subject of embarrassment and difficulties for several generations ... Thus are nations tormented about trifles'

from T. W. Körner via Somerset de Chair, editor.
Napoleon's Memoirs, Faber and Faber, London, 1958.

With a Little Help From My Friends

'I am a mathematician to this extent: I can follow triple integrals if they are done slowly on a large blackboard by a personal friend.'

J. W. McReynolds

Bullshit (from cows) Really is Important

"Science is full of **cowdung**, that irreverent shorthand for "the conventional wisdom of the dominant group!"

Bernard Wood
Opinion Books, *New Scientist*, 1 July 2000

Nature's Websites - a Sign of the Times

James Morley relates that his two-and-a-half-year-old son came in from the garden at the weekend and announced: "I've just seen a spider on his website."

Opinion Feedback,
New Scientist, 24 June 2000

Challenging the Mathematics Community

The contemporary mathematics community as a whole faces a critical challenge: to foster more interaction at every level of teaching and practice, and to widen the channels of communication with the students who will eventually replace us and extend our work into the future.

adapted from Phillip A. Griffiths
Mathematics at the Turn of the Millennium
in *AMM*, January 2000

The Codabar Check

Every credit card is uniquely identified by a 16-digit number. The first 15 digits are assigned by the bank issuing the card. The last digit (called the check digit) is determined by a formula which enables a computer to check that the number has been correctly entered when the card is used in a transaction.

All the major credit card companies use a system called Codabar to assign the check digit. This is how it works.

Suppose that the first 15 digits are

3141 5926 5358 979

Add the digits in the odd positions and double the total.

$$(3 + 4 + 5 + 2 + 5 + 5 + 9 + 9) \times 2 = 84$$

Add the digits in the even positions

$$1 + 1 + 9 + 6 + 3 + 8 + 7 = 35$$

Count the number of digits in the odd positions which are greater than 4. The numbers in this example are 5, 5, 5, 9 and 9, that is altogether 5 numbers.

Now add together the three numbers obtained in this way to get

$$84 + 35 + 5 = 124$$

The check digit is the number needed to bring this total up to the next multiple of 10. What should this be in our example? Yes, the required check digit is 6.

The Codabar system is one of the most efficient error detection methods. It picks up 100% of all single digit errors, and most other common errors such as switching two adjacent digits.

Have a look at some credit cards for yourself and you will find that the last digit is given by the formula described here. Test it to see if it picks up errors likely to be made in transcribing the number.

John Webb, University of Cape Town
From the NRICH Website at
www.nrich.maths.org.uk, September 2000

PROBLEM

Remember that we include a coding system which attempts to indicate in terms of Year levels the suitability range for each item. Thus 6 - 8 suggests an item accessible to students from Year 6 to Year 8.

How Obvious Is It?

7 - 12

How obvious is it that in a set of objects not all of the same colour and not all of the same size there will be at least two which differ in both colour and size? Can you explain?

This problem comes from a piece with this title by Hazel Perfect in *A Mathematical Spectrum Miscellany*, published by the Applied Probability Trust, Sheffield, 2000.

THUMBNAIL (and longer) REVIEWS

Readers are welcome to contribute to this section. Reviews can cover books, periodicals, videos, software, CD ROMs, calculators, mathematical models and equipment, posters, etc.

Relearning Mathematics
A Different Third R - Radical Maths Volume 1
Marilyn Frankenstein
Published by Free Association Books, 1989.

Note: Professor Frankenstein will be one of the three keynote speakers at AAMT 2001 next January in Canberra. I wrote the following piece when I first came across the book in 1991.

This is a powerful book intended mainly for adult learners returning to study. It provides an alternative, critical maths literacy founded on setting mathematics learning in real social and political contexts that take account of gender, race, class, history, politics and culture. Frankenstein's approach seeks to empower the learner by demonstrating that she already knows plenty; by affirming that mathematical literacy is a normal condition; by discussing the learning process and by encouraging self-evaluation, questioning and reflection; by lifting the lid on mathematical abstraction, pattern and explanation; by incorporating real-life subject matter and by painting in critical background knowledge; by encouraging the searching out of information and by demystifying the processes of drawing conclusions from quantitative data; by integrating literacy and mathematics; by showing that mathematics relates to every level of existence; and by demonstrating that mathematical understanding is essential for intelligent citizenship.

There are two forewords, the first by Henry A. Giroux and the second by Europe Singh, and an afterword by Ubiratan D'Ambrosio. All three of these reinforce the message that human beings have created mathematics so that we can better interpret and act on the world. In D'Ambrosio's words "Decisive for the political empowerment of individuals, mathematics, like reading and writing, must be treated as historical, social and political."

Part One of the book, covering about sixty pages is titled Mathematics: anxiety, anger, accomplishment. It has sections on misconceptions about learning mathematics, mathematics anger, the point of using calculators, keeping a journal (very valuable), working with others, multicultural considerations, and self-evaluation. Part Two, titled The meaning of numbers and variables, occupies just under two hundred pages and deals with fractions, decimals, percentages, whole numbers, signed numbers, measurement and comparison. A second volume will presumably extend the coverage to geometry and further abstractions involving numbers and symbols. The book ends with an appendix on critical maths, a few pages of notes, and a useful list of references. Unfortunately it lacks an index.

Frankenstein's teaching approach assumes students are serious about their learning and want to take responsibility for it. Her material neatly balances practice, process, application and critical awareness. She has created a unique and essential resource for any teacher of mathematics who works with adult learners and one which has much to offer teachers of younger students too.

Peter Enge

Notable Women In Mathematics:
A Biographical Dictionary
edited by Charlene Morrow and Teri Perl

Until recently, *Women of Mathematics: A Biobibliographic Sourcebook*, 1987, edited by Louise Grinstein and Paul Campbell (also from Greenwood Press), and Teri Perl's *Women, Numbers and Dreams* were practically the only two reference works on women in mathematics. This new book complements the other two by emphasizing the human element. The result is a valuable book for and about women in mathematics at a level suitable for high school and early college students that celebrates the diversity present in the community of women mathematicians.

The goal of the editors, stated on page xv, is admirable: to encourage more girls into mathematics and to spark enthusiasm for the field in all students. To this end, they have collected biographical information on 59 women mathematicians, spanning the last 2000 years. The emphasis, however, is on contemporary women mathematicians: of the 59, 40 are currently living and working in the field of mathematics or mathematics education in the US or in Europe. Much of this material comes from personal interviews, and thus goes beyond each woman's mathematics to discuss the person behind the mathematics.

The essays are a delight to read. Each gives a very human and encompassing picture of the whole woman - as mathematician, teacher, mother, wife or partner. Each one illustrates the path followed, not just professionally but also personally. Each woman's mathematical contributions are explained in terms that are easily understandable by a high school or college student interested in what a mathematics career may be like. This book provides an excellent complement and companion volume to the recently reviewed *Women in Mathematics: The Addition of Difference,* by Claudia Henrion.

The majority of the women featured are or were academics, but some developed careers in government and industry. The tension between career and marriages, husbands, partners, children and other family responsibilities is a central part of their experience. These are real people, trying to find a balance between lives and careers just as the rest of us struggle to do. Many of the stories can serve to provide encouragement for young women contemplating a career in the mathematical sciences, and also for those who just wonder what it might be like.

The obstacles of gender and racial discrimination that some of these women faced in pursuing a mathematical career are vividly portrayed. For example, Lenore Blum was not admitted to MIT for her undergraduate work because there were only 20 dormitory beds available for women. She was married at 18 and her mother cared for her son while she worked on her thesis. More than one woman (McDuff, Kovalevsky) wrote two or more dissertations. Vivienne Malone-Mayes could not join her classmates and advisor in discussing mathematics over coffee because she was black. She was denied a teaching assistantship, also because she was black. She could not enrol in one professor's classes because he refused to teach blacks and furthermore believed that it was a waste of taxpayer's money to educate women! And in case we think we need no further vigilance, Joan Sterling Langdon, a student of Mary Gray's, was one of only three African American women who earned a Ph.D. in mathematics or mathematics education in the US in 1989.

Many of these women were married to mathematicians or scientists. There are several discussions of the two body problem of finding a position for a husband and wife pair at the same institution, and how difficult it was because of nepotism laws.

The women in academic positions clearly excelled in the three components of such a career -- teaching, research and service -- and the contributing authors provide many examples of these efforts. Many of them rose to the position of department chair and most were instrumental in developing policy for mathematics departments. Many of them were advocates for women, blacks, minorities of any kind. Unequivocally, one is left with the conviction that women have done, can do, and continue to do mathematics.

Despite their success, self-doubt seems to be a common thread running through the stories of these women. Many of them wondered whether they had what it would take to be successful, sometimes even after they had earned a Ph.D. and were holding an academic position (see, for example, Dusa McDuff). Leone Burton, a math educator, comments that as an undergraduate she didn't have even one woman teacher who might serve as a role model or as a source of encouragement. In contrast, others (see Michelson) report getting continual feedback from their professors as well as encouragement to continue on to graduate school. It is clear that these women welcomed any encouragement, however small, and whenever encouragement was received it went a very long way towards keeping them going in the face of adversity and lack of confidence. Rhonda Hughes says of her college mathematics experience: "It took so little to encourage me, but it was so important."

There is one outright error, on page 39: Émilie du Châtelet lived in the 18th, not the 17th, century. There is also a very misleading statement on page 65 about the proof of Fermat's Last Theorem. The author suggests that Andrew Wiles' proof was only possible with the use of computers. This is not the case. Wiles' proof was in fact a traditional "pencil and paper" proof, although the fact that people could check the Shimura-Taniyama conjecture computationally was an important part of the reason it came to be accepted as likely to be true.

Overall, I would recommend *Notable Women In Mathematics* to high school teachers of mathematics, high school counsellors, high school students interested in mathematics, and also to college professors and students of mathematics looking for a connection to the diverse community of women mathematicians. This book is a wonderful source of role models for young women who may be attracted to mathematics but who may at the same time be repelled by the stereotypical image of the lone, slightly mad, probably male, certainly eccentric mathematician.

Libby Krussel
MAA Online

Mathematical Problems and Proofs: Combinatorics, Number Theory, and Geometry
by Branislav Kisacanin
Published by

When I first set out to read this book in preparation for this review, I was looking for a textbook for an undergraduate discrete mathematics course. It turns out that this book was not appropriate for the course I had in mind. But, to be fair, it was not written as an undergraduate textbook. In the words of the author:

This book is written for those who enjoy seeing mathematical formulas and ideas, interesting problems, and elegant solutions.

More specifically it is written for talented high-school students who are hungry for more mathematics and undergraduates who would like to see illustrations of abstract mathematical concepts and to learn a bit about their historical origin.

It is written with that hope that many readers will learn how to read mathematical literature in general.

There is no question that this book will be well received by the target audience. The book contains an engaging mix of topics and includes some topics that are often not contained in a high school or college mathematics curriculum, but that build a student's appreciation and love for mathematics. Some examples are: proofs of the Pythagorean theorem and a discussion of Pythagorean triples; a discussion of the arithmetic, geometric, and harmonic means and the relationships among them; and several approaches to the Towers of Hanoi problem.

The book is divided into 4 chapters and a set of Appendixes. There is also a Key to Symbols that will be particularly useful to those encountering notations for the first time. The chapters each cover one general subject area: Chapter 1 deals with Set Theory, Chapter 2 with Combinatorics, Chapter 3 treats Number Theory, and Chapter 4 focuses on Geometry. The four appendixes deal with Mathematical Induction, Important Mathematical Constants, Great Mathematicians, and the Greek Alphabet. Following those are a nice set of references and a thorough index.

Each of the chapters is divided into three or four sections of content followed by a selection of problems. The examples/ problems include detailed solutions. A feature that I find particularly nice is the fact that more than one solution for a problem is often provided. Too often mathematics books leave students thinking that there is only one right way to solve a problem. Students often stumble onto a clever way of solving something and are left to wonder whether their technique is valid. (Unfortunately there are some teachers who actively discourage using any method other than the one being taught in the current section of the official class text.) This book makes it clear that there are many ways of solving some problems. By seeing several solutions the students will learn not only this fact but also learn additional techniques which will be useful in solving other problems. According to the author, the problems are designed to illustrate theorems and ideas and to develop the reader's problem-solving ability and sense for elegant solutions. I think that he has done a good job at selecting problems and providing solutions and explanations which succeed at this task. In order to pique the reader's interest in this book, I will describe some of the content in detail.

The chapter on Set Theory is brief (only 16 pages) but manages to pack in a lot of material without overwhelming the reader. In addition to a very complete set of definitions and properties, the author includes many examples. He proves that the square root of 2 is irrational and that the irrational numbers are uncountable. He discusses equivalence classes and touches on algebraic and transcendental numbers. In short, he covers a selection of topics that will grab hold of an eager student's interest.

The chapter on Combinatorics is considerably longer at 52 pages. Again the author has chosen topics which do an excellent job of covering the basics of combinatorics as well as capturing the interest of the curious reader. Topics range from basic counting problems to discussions of Euler's phi function, generating functions (including material on the Fibonacci numbers), and probability distributions used in statistical physics (Maxwell-Boltzmann statistics, Bose-Einstein statistics, and Fermi-Dirac statistics)! Quite a large range of topics, but, once again, dealt with in a way which does not overwhelm the reader.

The chapters on Number Theory (42 pages) and Geometry (43 pages) again cover the basics in a clear way and with a nice selection of supporting examples and problems. Once again the topics which go beyond the basics are chosen to reach out to the reader and show him or her what it is that makes mathematics fun and interesting.

I was particularly impressed with the Appendix on Mathematical Induction. At 22 pages, it is longer than the entire chapter on Set Theory and it includes numerous well chosen examples and problems.

There are some peculiarities to this book. For example, the pigeonhole principle is referred to as Dirichlet's principle. While this is an appropriate name for the principle, it is almost universally referred to (at least in my experience) as the pigeonhole principle. I would think that at the very least the student should be introduced to the common name. But apart from a few rather minor complaints like this one, the book is excellent. I would highly recommend it for mathematics departments, particularly at the high school level. This book belongs in the hands of eager high school and undergraduate level mathematics enthusiasts who will benefit from its range of topics and its large set of over 150 thoroughly solved examples and problems.

Carl D. Mueller
MAA Online

SOLUTION TO THE PROBLEM

Select objects from the set one by one and tally them by colour and shape in the corresponding cell in the table, with colour 1 the colour of the first object selected and shape 1 its shape. Since the objects are not all the same colour there must be a non-zero entry in each column. If these non-zero entries are in different rows we are done, otherwise there must be a third non-zero entry in the other row, so either way we have at least two objects differing in both colour and size.

	Colour 1	Not colour 1
Shape 1		
Not shape 1		