CLASSROOM CAPSULES

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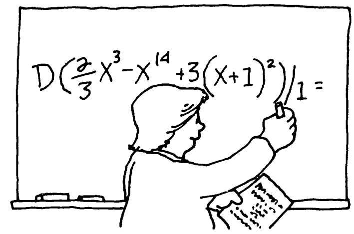

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia, Department of Mathematics, Miami University, Hamilton, OH 45011.

The Best Shape for a Tin Can*

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Some time ago, I came across a book intended to popularize mathematics, whose last chapter dealt with the calculus of one variable. Its final section, evidently intended to climax the whole work, solved the problem of designing the proportions of a tin can so as to obtain the greatest volume out of a given amount of material. The well-known solution is of course that the material used is proportional to

$$M = 2\pi r^2 + 2\pi r h, \tag{1}$$

whereas the volume is

$$V = \pi r^2 h, \tag{2}$$

so that

$$h = \frac{V}{\pi r^2}. \tag{3}$$

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Inserting (3) into (1) and setting $dM/dr = 0$ gives

$$4\pi r - \frac{2V}{r^2} = 0, \tag{4}$$

or, in view of (2)

$$h = 2r. \tag{5}$$

In other words, the most economical shape has its height equal to its diameter. The authors then drew attention to the fact that most cans are not 'square' and sought to explain the discrepancy. They concluded that tradition and design fashion must count for more than rational thought in the commercial world. Their parting message to the reader was to the effect that intellectual beauty was its own reward.

Such a patrician view of mathematics is, these days, a luxury that few can afford. The irony is that the chosen example, and the discrepancy between the mathematical model and the real world, actually illustrate rather nicely the true character and value of applied mathematics.

Consider first that when the lid and base of the can are cut from sheet there must be wastage, which is presumably returned for recycling, but has little value to the can makers. If we suppose that the sheet is divided first into squares of sides $2r$, and that one circle is cut from each square, equation (1) should be replaced by

$$M = 8r^2 + 2\pi rh, \tag{6}$$

leading to

$$\frac{h}{r} = \frac{8}{\pi} \simeq 2.55. \tag{7}$$

A better strategy (from a mathematical viewpoint, anyway) would be to divide the sheet into a honeycomb of hexagons. Neglecting the waste at the edge of the sheet, we find

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \simeq 2.21. \tag{8}$$

Although this may be interesting, neither (7) nor (8) describe very well the usual proportions of a can. We are still forgetting things. For example, examination of a real tin can shows that the top and bottom are formed from discs somewhat larger than r in radius, which are then shaped over the ends. Allowing for this would increase (h/r), as would any extra costs associated with forming a lid, or making the lid of thicker material. Also importantly, the cost of a can needs to include its fabrication as well as its materials. If the most costly operation consists of joining the side and two rims of the can, the total cost is proportional, with most economical cutting, to

$$c = 4\sqrt{3}\,r^2 + 2\pi rh + K(4\pi r + h), \tag{9}$$

where K is the length that can be joined for the price of buying unit area of material (with this definition (9) is dimensionally consistent). Then repeating the earlier maneuvers leads to

$$4\sqrt{3}\,r - \frac{V}{r^2} + 2\pi K - \frac{KV}{\pi r^3} = 0 \tag{10}$$

as the condition to be satisfied by an optimum design, together of course with (3).

Extracting the information from (10) requires some ingenuity. Dimensional analysis suggests a relationship between (h/r) and (V/K^3) and indeed (10) will yield it. After dividing by r, the terms are regrouped as

$$4\sqrt{3} - \frac{V}{r}\left(\frac{1}{r}\right)^2 + \frac{2\pi K}{r^{1/3}}\left(\frac{1}{r}\right)^{2/3} - \frac{KV}{\pi r^{4/3}}\left(\frac{1}{r}\right)^{8/3} = 0,$$

and the terms in $(1/r)$ can be substituted by $(\pi h/V)^{1/2}$, leading to

$$4\sqrt{3} - \frac{V}{r}\left(\frac{\pi h}{V}\right) + \frac{2\pi K}{r^{1/3}}\left(\frac{\pi h}{V}\right)^{1/3} - \frac{KV}{\pi r^{4/3}}\left(\frac{\pi h}{V}\right)^{4/3} = 0.$$

From this, finally,

$$\left(4\sqrt{3} - \frac{\pi h}{r}\right) + \left(\frac{\pi K^3}{V}\right)^{1/3}\left(\frac{h}{r}\right)^{1/3}\left(2\pi - \frac{h}{r}\right) = 0. \tag{11}$$

This can be thought of as a quartic for $(h/r)^{1/3}$, with K^3/V as a parameter, but several things are immediately clear. First, if joining is cheap (K small) or the can is large (V large) then we have the original design ($h/r = 4\sqrt{3}/\pi$). In the opposite cases where joining is expensive, or the can is small, we find $h/r = 2\pi$. This corresponds to manufacturing costs greatly exceeding material costs. Also, for any other situation there is by Rolle's theorem a value of (h/r) between these limits that satisfies (11).

A graph of the complete relationship (11) is easily made by inverting it to read

$$\frac{V}{K^3} = \pi\left(\frac{h}{r}\right)\left[\frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}\right]^3, \tag{12}$$

which allows Figure 1 to be plotted.

The predicted trend, that big cans should be nearly square, whereas small cans should be tall and thin, can be verified in a supermarket. Compare, for example, cans of marmalade oranges with cans of cocktail olives. However there are cans that do not fit the trend. Sometimes there is an explanation deriving from the nature of the product (pineapple rings, for example). Very small cans are often squarer than one would expect, perhaps for convenience of handling the tin opener. Convenience would also be a consideration for any can designed to be drunk from.

To summarize, the failure of the original model to predict the real shape of a tin can arises from its being a very naive model. More complete models, still within the range of school mathematics, began to reveal the real issues. Without complicating the analysis, many other questions could be explored. Would the argument be affected if the sheet material is only available in standard sizes? Or if we knew

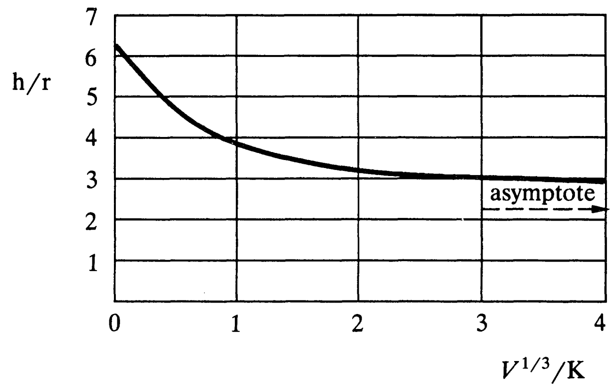

Figure 1

how the can-making machine actually worked? Are we designing to a given volume? Or to a given weight of contents? How are the cans stacked for transport? How much does it cost to be slightly off the optimal proportions? Is this offset by any other saving?

There is plenty of scope here for developing a true appreciation of how mathematics contributes to technology.

The Curious 1/3

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The United States Postal Service will accept for delivery only packages that conform to this rule: the length plus the girth must not exceed 108 inches. This rule generates a number of maximization problems in calculus texts [**1**, p. 260], [**2**, p. 240], [**3**, p. 220]. The answers to these problems contain an interesting surprise.

To generalize slightly we may replace 108 by any positive constant c. The typical textbook problem asks for the maximum volume of a right cylindrical package when the cross section of the package is a square or when it is a circle. The length of the box of maximum volume is, in both cases, $c/3$.

In a recent class I asked the students to try using equilateral triangles for the cross section. The length of the box of maximum volume was still $c/3$. The reader might want to try other shapes, perhaps an isosceles right triangle.

Is this just a strange coincidence? Or is the length of the box of maximum volume actually independent of the shape of the cross section?

Suppose we decide on a (reasonable) shape for the cross section of the box. Consider one example of that shape with perimeter P and area A. We may use a "magnification factor" x to describe all similar shapes, which will have perimeter Px and area Ax^2. (It should be easy to convince students of the existence of such a magnification factor, at least in the case of figures that can be decomposed into a finite number of triangles. For more complicated figures and more advanced students an argument using line integrals for the perimeter and area should be convincing.)

Let z be the length of the box. We seek the maximum volume $V = Ax^2z$ subject to the restriction $Px + z = c$. This constraint can be solved for x to yield $V = (A/P^2)z(c-z)^2$ for $0 \le z \le c$.