The poetry of mathematics

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I read in a book by W.W. Sawyer that to be complete, a mathematician needs to be something of a poet. The author was quoting the 19th century German mathematician Karl Weierstrass.

Sawyer was setting out to write a book about nurturing young mathematicians. The quote appears right at the beginning of chapter one. The book is called Prelude to mathematics.

Nurturing is exactly what we are doing, as teachers of mathematics. I hope to give some hints about where the poetry might appear in this enterprise.

You will have heard of the reSolve – Maths by Inquiry project aimed at promoting of a spirit of inquiry in classrooms. For this purpose, a collection of classroom materials is being produced under the guidance of the following protocol:

- reSolve mathematics is purposeful
- reSolve tasks are challenging yet accessible; and
- reSolve classrooms have a knowledge-building culture

However, it is not my purpose to say much about the reSolve project. Instead, I want to talk about an experience of helping to develop just one of the classroom tasks. The purpose of this particular task was to show the power of algebra to encapsulate whole classes of arithmetic statements in a generalised form.

The task is not original. Indeed, there is a YouTube video of Brady Haran demonstrating it at https://www.youtube.com/watch?v=CWhcUea5GNc. And other versions can be found.

It goes like this:

[DEMONSTRATE as with a class]

The teacher invites students to think of two numbers, write them down one below the other on a whiteboard and to add them, putting the sum on a third line. Then, the second number is added to the third and the sum is appended to the list. The process continues until the list contains ten numbers.

Somehow, the teacher is able to announce the sum of the ten numbers in the list before they have all been written down. What is the trick? Do a few examples. Look at the sums and look at the numbers in the lists. There must be a hidden relationship.

In the classroom, someone might spot the fact that the sum appears to be 11 times the seventh number. This discovery has, at first, the status of a conjecture and we are going to need some algebra to confirm that it is true in all cases. We construct the same list as before but starting with *a* and *b* to stand for a general choice of numbers. As well, we construct a parallel list of cumulative sums so that, at the end, it is clear that the sum of the ten numbers is 11 times the seventh number, no matter what the first two chosen numbers were.

n	Addition Chain	Cumulative Sums
1	a	a
2	b	$a+b$
3	$a+b$	$2a+2b$
4	$a+2b$	$3a+4b$
5	$2a+3b$	$5a+7b$
6	$3a+5b$	$8a+12b$
7	$5a+8b$	$13a+20b$
8	$8a+13b$	$21a+33b$
9	$13a+21b$	$34a+54b$
10	$21a+34b$	$55a+88b$

This is all good. If the students have been captivated by this, the main purpose of the lesson has been achieved. But, there is more, as Steve Thornton and I discovered when we were looking at this activity.

We observed other connections between the addition chain numbers and the cumulative sums and wondered whether extending the lists might reveal anything interesting.

If the sequence of numbers in the first list is given the notation t_n and those in the cumulative sum list are labelled s_n, we can write the following after looking at the lists as far as they go:

$$s_1 = t_1$$
$$s_2 = t_3$$
$$s_3 = 2t_3$$
$$s_6 = 4t_5$$
$$s_{10} = 11t_7$$

While the first result is obvious, it is curious that, if we ignore the third result, the subscripts on the addition chain numbers t_i in the relations found above are consecutive odd numbers while there is a hint that the subscripts on the cumulative sums s_i, after s_2, might increase in jumps of 4.

We checked and found that this was indeed the case.

$$s_{14} = 29t_9$$

$$s_{18} = 76t_{11}$$
$$s_{22} = 199t_{13}$$

We appeared to have a sequence of coefficients:

$$1, 4, 11, 29, 76, 199, \ldots$$

This was new and exciting for us. Naturally, we wanted to know whether there would be further terms of this sequence and how we might find them. First, we needed a rule that would generate these terms and then, we needed to prove that the rule would continue to be correct if the table were to be extended indefinitely. At first, we were stumped on both counts.

This is not the place to go through the mathematics in detail. You can study the handout for that if you wish, (attached). I want to outline just the landmarks along the way and to think about the poetry.

An internet search of the Online Encyclopedia of Integer Sequences gave us a named sequence that corresponded with the terms we had found but this did not guarantee that our sequence would continue in the same way as the one in the encyclopaedia.

At this point, a certain amount of poetry begins to enter the story. There is a joke that I heard many years ago.

Imagine, it is dark and under a streetlight, a person is apparently looking for something lost. You approach and offer to help. You ask, 'Where did you lose it?' 'Somewhere over there', indicates the searcher. 'Then, why are you looking here?' you ask, to which the other person explains, '…because the light is better '.

A mathematician is the person looking for something in the dark. When faced with a difficult problem it might be better, sometimes, to look a little bit away from it, at things that are known and understood, in the hope that new connections will come to light that relate to what is yet to be found.

So, what did I already know that might conceivably have anything to do with this problem?

1. We had been designing a lesson about the power of algebra to make general statements. So, perhaps the first step should be to write down the relation that we were investigating in a concise way. Using c_n for the *n*th coefficient, we can see from looking at the table of addition chain numbers and cumulative sums that

$c_1 t_3 = s_2$	$1(a + b) = a + b$
$c_2 t_5 = s_6$	$4(2a + 3b) = 8a + 12b$
$c_3 t_7 = s_{10}$	$11(5a + 8b) = 55a + 88b$
$c_4 t_9 = s_{14}$	$29(13a + 21b) = 377a + 609b$

In general, these relations are expressed by

$$c_n t_{2n+1} = s_{4n-2}$$

2. The addition chain is clearly a Fibonacci-like object. I knew that there exist various relations between terms of the Fibonacci sequence and while I could not immediately recite them exactly, I probably could recover one or more of them if required.

3. The Online Encyclopedia of Integer Sequences told us that if we started a Fibonacci-like process with the numbers $\{2, 1, 3, 4, 7, 11, \ldots\}$ and then looked at every second term, we would have our sequence, $\{1, 4, 11, 29, \ldots\}$.

   You might be able to spot a simpler way of getting the next term from the ones given. It turns out that if we start with $\{1, 4, \ldots\}$, then the next terms are given by

$$c_{n+1} = 3c_n - c_{n-1}$$

4. It seemed that it might be possible to say something useful about the apparent relation between the addition chain terms, the progressive sums and the sequence of coefficients, if all of them were somehow to be expressed in terms of the Fibonacci sequence. Fortunately, this proved to be possible.

$$\begin{aligned} t_i &= F_{i-2}a + F_{i-1}b \\ s_i &= F_i a + (F_{i+1} - 1)b \\ c_i &= 2F_{2i} - F_{2i-1} \end{aligned}$$

5. From here on, the proof involved algebraic manipulation and some facts about the Fibonacci sequence. We can now rest, knowing that the pattern Steve and I observed does indeed go on forever and the sequence of coefficients we found is the right one.

Did you notice any poetry?

One might speak of the pleasure of crafting an elegant chain of reasoning in mathematics and liken this to the experience of creating well-organised images with words, as in written poetry. This may describe the somewhat vague aesthetic connection between the two disciplines.

Poetry, it seems to me, can tell a story or it can explore a static situation of some kind. In either case, it presents and combines images that are likely to be familiar to the reader. In a broad sense, it might be said that poetry is the action of recalling familiar ideas and images and making new and striking connections between them.

Could this be what Karl Weierstrass had in mind when he claimed that '… to be complete, a mathematician needs to be something of a poet'?

My picture of a mathematician is someone who draws together ideas from a wide experience of mathematics in creative ways in order to solve problems. New discoveries in mathematics, I suspect, tend not to come from random tinkering or from slavishly following routine procedures but from minds highly experienced in the discipline drawing on a variety of resources.

Thus, in nurturing new mathematicians, it would seem to be a good idea to teach some routine procedures, by all means, but to place these in a much richer context of mathematical ideas. Mathematics is not so much about procedures as it is about imagination, and this makes it something of an art.

I would like to finish with another example illustrating the undesirability of reaching immediately for the obvious routine tool rather than looking a little bit away, where the light might be better.

You may have seen a challenge to solve an equation like

$$\sqrt{x+37}+\sqrt{x}=37$$

If you were like me, you would be thinking about how to get rid of the square roots. Perhaps we should square both sides and maybe do it again because squaring once will not quite do it. We might end up with a quadratic but we know how to deal with those.

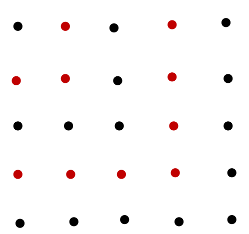
But, why would anyone issue a challenge like this one? Could there be a trick? Is there something special about the number 37? Is there a more elegant way to get at the solution?

The writer W.W. Sawyer in the book I mentioned previously touches on this issue. This problem surely did not just pop up out of nowhere. It must have arisen in the context of some other piece of research now conveniently hidden for the sake of posing a challenge.

Has anyone seen a strategy better than the one just outlined?

One thing that would make the square roots go away would be if both x and $x+37$ were perfect squares. Are there two squares that differ by 37?

And here we might recall a picture something like this



which tells us that square numbers are formed by adding consecutive odd numbers. We should make $x = 1 + 3 + 5 + \cdots + 35$, which must be a square. Then $x + 37$ will be the next square.

If the last layer in the square diagram has 35 dots in it, it must form a square measuring 18×18. So, $x = 18^2$. Then the next square is 19×19 which should be $18 \times 18 + 37$ and, of course, $18 + 19 = 37$.

Something more falls out of this way of looking at the given problem. We see that any equation of the form

$$\sqrt{x+n} + \sqrt{x} = n$$

where n is an odd integer, has an integer solution, namely $x = \left(\frac{n-1}{2}\right)^2$. There was nothing special about the number 37 after all, apart from the fact that it is an odd integer.

Thank you.

Mathematical details

It will be necessary to prove that for positive integers n,

$$c_n t_{2n+1} = s_{4n-2}$$

To do this, it will be helpful if each term in this proposed equation can be expressed in terms of Fibonacci numbers, F_i.

We have, for $i > 2$,

$t_i = F_{i-2}a + F_{i-1}b$ and

$s_i = F_i a + (F_{i+1} - 1)b$.

By inspection, it appears that $c_i = 2F_{2i} - F_{2i-1}$. This is proved by induction. It is true when $i = 3$ and we suppose it is true that $c_k = 2F_{2k} - F_{2k-1}$ for a positive integer $k > 3$. We have

$$\begin{aligned} c_{k+1} &= 3c_k - c_{k-1} \\ &= 3(2F_{2k} - F_{2k-1}) - \left(2F_{2(k-1)} - F_{2(k-1)-1}\right) \end{aligned}$$

And after several applications of the fact that $F_{n-1} = F_{n+1} - F_n$, we find that $c_{k+1} = 2F_{2k+2} - F_{2k+1}$ as required.

Thus, to prove that

$$c_n t_{2n+1} = s_{4n-2}$$

we have to prove that

$$(2F_{2n} - F_{2n-1})(F_{2n-1}a + F_{2n}b) = F_{4n-2}a + (F_{4n-1} - 1)b$$

for all values of a and b.

This can only be the case if both the following equations hold:

$$(2F_{2n} - F_{2n-1})F_{2n-1} = F_{4n-2}$$

and

$$(2F_{2n} - F_{2n-1})F_{2n} = F_{4n-1} - 1$$

We will need three lemmas about Fibonacci numbers:

(1) $F_{2k} = F_{k-1}F_k + F_k F_{k+1}$

(2) $F_{2k+1} = F_{k+2}F_{k+1} - F_k F_{k-1}$

(3) $F_{k-1}F_{k+1} = F_k^2 + (-1)^k$

Using (1), we have

$$\begin{aligned} F_{4n-2} &= F_{2(2n-1)} \\ &= F_{2n-2}F_{2n-1} + F_{2n-1}F_{2n} \\ &= (F_{2n} - F_{2n-1})F_{2n-1} + F_{2n-1}F_{2n} \\ &= F_{2n-1}(2F_{2n} - F_{2n-1}) \end{aligned}$$

as required.

Using (2), we have

$$\begin{aligned} F_{4n-1} - 1 &= F_{2(2n-1)+1} - 1 \\ &= F_{2n+1}F_{2n} - F_{2n-1}F_{2n-2} - 1 \\ &= (F_{2n} + F_{2n-1})F_{2n} - F_{2n-1}(F_{2n} - F_{2n-1}) - 1 \\ &= F_{2n}^2 + F_{2n-1}^2 + (-1)^{2n-1} \end{aligned}$$

Then, using (3),

$$\begin{aligned} F_{4n-1} - 1 &= F_{2n}^2 + F_{2n-2}F_{2n} \\ &= F_{2n}(F_{2n} + F_{2n-2}) \\ &= F_{2n}(2F_{2n} + F_{2n-1}) \end{aligned}$$

as required.

□

For completeness, we should prove the three lemmas concerning Fibonacci numbers that were used in the proof. This can be done using induction arguments that we leave to the reader.