Population Modelling
I. Exponential Growth

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Contents

Population Modelling Series

Population Modelling I. Exponential Growth looks at simple exponential-growth models and problems; suitable for Years 7–9.
Population Modelling II. Non-Exponential Models looks at other types of population models, both continuous and discrete; suitable for Years 10–12.
Population Modelling III. SIR Epidemic Model works through a classic model of the spread of an epidemic; suitable for good students in Year 12 .

1 Introduction to Population Modelling

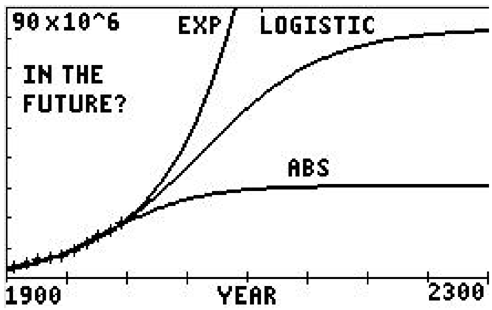
When mathematicians talk about playing with a model, chances are they don't mean a model plane or boat. They are probably talking about a mathematical model — a set of equations that describe in mathematics how a particular system works. There are mathematical models for many things, such as the planets revolving about the sun, heating iron ore in a blast furnace, pollution in a lake, how prices vary on the stock exchange, the spread of diseases and how populations (people, animals, bacteria, viruses, etc) change with time.

Population modelling started a long time ago, and one of the earliest modellers was Fibonacci (1170–1250). In his book *Liber abaci*, he modelled a rabbit population, starting with one pair of baby rabbits. If each adult pair of rabbits produces only one pair of baby rabbits each month, and if baby rabbits take one month to become adults, the numbers of pairs of rabbits in successive months are given by the famous Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, and so on. The next number is found by adding the previous two numbers. Fibonacci numbers are also found elsewhere in Nature. If you look at a pine cone, you will find the 'petals' spiral in two directions. The number of petals it takes to get once around is almost always a Fibonacci number. The same thing occurs in pineapples, sunflowers and many other flowers.

Much later, Thomas Malthus (1766–1834) in England startled the world by predicting that food would run out sometime in the future, because of the rapid increase in the human population. Based on the data he had at the time, Malthus predicted that the world population would increase exponentially, doubling every 40 years, thereby increasing at a faster and faster rate. (Forty years is the current doubling time of the world's population.) If you start with the number 1 and keep doubling it, you will see an example of exponential growth.

The models of Fibonacci, Malthus and some other scientists all predict that the population will grow faster and faster. This is an alarming prospect, but does not seem to happen in experiments performed when there are limited resources, such as food and space to live in. Experiments with small animals and fungi in the laboratory, and with larger animals in fenced areas in the field show that as the resources start to run out, the reproduction rate reduces and the rate of growth slows down. The Belgian scientist Pierre Verhulst (1804–1849) while at the Belgian military school, the Ecole Royale Militaire, developed a model, called the logistic model, which took into account these observations. He introduced the idea of a 'carrying capacity' or maximum sustainable population that the environment will support.

We can illustrate the Malthus exponential model and the Verhulst logistic model by looking at the population of Australia since 1900. The small dark crosses on the graph below show the Australian Bureau of Statistics figures for the number of people in Australia (in millions) up until 1996. If we model these data with an exponential curve, we get the top curve in the figure. The middle curve is the logistic model. Both these curves fit the population numbers up to the present time well, but predict quite different future populations.



According to the exponential (Malthus) model, the population will continue to grow at a faster and faster rate, with a predicted population of about 109 million people in the year 2100, and about 587 million people in 2200. The logistic (Verhulst) model predicts that the population will keep on growing, but at a slower and slower rate; the predicted population in 2100 is about 57 million people, and the population would level off eventually at about 83 million people.

The Australian Bureau of Statistics (ABS) uses a mathematical model to predict the population of Australia well into the future to assist in planning for the number of people who will be living here. The predictions of their model are shown as the bottom curve in the figure. It has the shape of a logistic curve, but levels out much faster than the middle curve, predicting a population in 2100 of about 30 million people, and a maximum population of about 31 million.

Prediction is one powerful aspect of a mathematical model. By putting in the numbers we know, such as for the Australian population, we can predict what a population will be in the future, according to our model. Of course, the accuracy of our predictions depends on how good our model is, that is how well it describes the phenomena that affect population growth.

Another important use of a population model is to predict what will happen to the population if something changes, for example if the birth rate drops, if the number of immigrants is decreased, or if, say in a war, many people die. Predicting changes in a population is particularly relevant to populations of animals, insects and plants which have become serious pests after being brought into Australia from overseas. These include rabbits (see the picture below), foxes, mice, cane toads and European carp among the animals, and prickly pear, Paterson's curse, salvinia, mimosa and scotch thistle, to name but a few of the plants. The populations of some of these have reached very high levels at times, causing serious problems for farmers and the environment.

How do we control such pests? Often there are a number of possible ways, but which one is best? Population models can be modified to include the effect of the release of a predator, the spread of a disease in the pest population, the effect of poisoning or some other control measure. It is then possible to use the mathematical models to predict what would happen to the population if the different control strategies were tried. The models can also be used to find the best way of carrying out a particular control measure. Sometimes the modelling is done together with small-scale experiments, but often only the model can be used because the experiments are too risky or too expensive.

Rabbits drinking at a waterhole before the introduction of the myxomatosis virus.

In using a population model, we put the starting conditions and parameters (number of animals, how quickly they breed, etc) into our equations and predict the population at some later time. What if we change the starting conditions only slightly? We will end up with nearly the same final answer, right? Not necessarily. In some models, for example a variation on the Verhulst logistic model, with particular parameters, we find that the population does not change steadily towards some ultimate population, as we saw in modelling the Australian population, but changes rapidly and unpredictably with time. We say the model exhibits chaos: it loses its ability to predict, because a small change in the starting conditions produces a large change in how the population varies with time.

2 Population Problems

Mathematically, the problems here are about *iteration* and about *exponential processes*.

Iteration is the process of carrying out the same operation over and over again. Let's take a simple example, that of multiplying by 2. Start with the number 1. Multiply it by 2 to give 2. Multiply the answer 2 by 2 again to give 4. Multiply 4 by 2 to give 8, and so on.

If you have a standard calculator, you may be able to do many of the calculations in the problems here just by pressing the [=] or [×] key. To do the calculation here, try this: press [1] [×] [2] [=], then just press [=] or [×] (depending on your calculator) to multiply by 2 each time. You'll have to keep count of how many times you have multiplied by 2. If this quick method doesn't work on your calculator, experiment to see what does.[1]

Iterating by multiplying by a constant (2 here) is an example of an *exponential process*. You may have heard the term exponential growth, which many people interpret to mean 'grow quickly'. But exponential growth has a precise mathematical meaning, and some interesting properties which we shall explore shortly.

Exponential iteration models a number of processes such as radioactive decay, population growth and absorption of light. If the constant we multiply by is larger than 1, we get exponential growth; if it is less than 1 (but greater than 0), we get exponential decay.

The use of scientific notation makes writing down our calculations much easier. For example, if we start with 5 and multiply it by 2 three times, we get $5\times2\times2\times2$, written as $5\times2^3=40$. 2^3 means three 2s multiplied together. If we multiply 5 by 2 ten times, we have $5\times2^{10}=5120$. 2^{10} means ten 2s multiplied together. Some calculators have an exponentiation key, usually [y^x] or [∧] so, to calculate 2^3, we would press [2] [y^x] [3] [=] or [2] [∧] [3] [enter].

2.1 Exponential Iteration

Write down the results of the first 10 iterations of multiplying by 2, starting with 1.

2.2 Lots and Lots of Bacteria

Bacteria multiply (increase in number) by dividing — into two. One type of bacterium, *Streptococcus exponentiae*, divides every minute. If we start with 1 bacterium, it divides into 2 bacteria after 1 minute. Each of these 2 bacteria divides after 1 more minute, and so on. *The number of bacteria grows exponentially.*

Make up a table with time in the first column and the number of bacteria in the second column. *How many bacteria are there after 10 minutes? after 20 minutes? after 1 hour? after n minutes? Why isn't the Earth covered metres deep in these bacteria?*

[1] On a TI-84/CE: [1] [enter] [×] [2] [enter] [enter] .... On a Casio 9860, use [EXE] instead of [enter].

2.3 Malthus and Exponential Growth

Thomas Robert Malthus (1766–1834) made some worrying predictions for the world population, and his name is often associated with the idea of exponentially growing populations. Look up Malthus to find out the details of his ideas. Why was he worried about the world's population?

Malthus looked at the United States population to try to verify his ideas. He concluded the growth was exponential. *From the numbers in the table below, can you tell if he was correct for the years until he died?*[2] *What about the population growth after about 1860?*

Year	**Population** (millions)	**Year**	**Population** (millions)	**Year**	**Population** (millions)
1790	3.90	1860	31.4	1930	123
1800	5.30	1870	38.6	1940	132
1810	7.20	1880	50.2	1950	151
1820	9.60	1890	62.9	1960	179
1830	12.9	1900	76.0	1970	203
1840	17.1	1910	92.0	1980	227
1850	23.2	1920	106	1990	249

Why might the populations not continue to increase exponentially?

Note: The TI-84/CE graphics-calculator programs in Section 5 make teachers' lives easier for this problem, and for modelling the Australian and world populations.

If students are to do the data plotting and curve fitting, they should do it manually rather than using a program. The data (contained in the programs) could be transferred from the teacher's calculator.

[2] *Hint*: If the growth is exponential, each population should be a constant multiple of the previous value. Try a multiplier of 1.35, meaning the population increased by 35% every 10 years. The numbers you obtain only need to be close to the actual numbers, not exactly the same.

2.4 Cane Toads

The Hawaiian cane toad *Bufo marinus* was introduced into Australia to control sugar-cane beetles. From the original 101 toads released in north Queensland in June 1935, the population grew rapidly and spread across the countryside. The table below shows the total land area of Australia colonised by cane toads for the years 1939 to 1974.

Year	**Area** ($1000\,\text{km}^2$)	**Year**	**Area** ($1000\,\text{km}^2$)
1939	33.8	1959	202
1944	55.8	1964	257
1949	73.6	1969	301
1954	138	1974	584

Is exponential growth a good model here? You can get a rough idea by the process we used for the Malthus data — finding ratios of successive values — but a plot of the data together with an exponential fit (graphics calculator required) will provide a better answer. *What is the exponential equation of best fit?*

Given that the area of Queensland is 1728 thousand km^2 *and the area of Australia is 7619 thousand km*^{2} *when, according to the exponential model, will (did) the cane toads colonise all of Queensland? all of Australia?*

The cane growers were warned by Walter Froggart, President of the New South Wales Naturalist Society, that the introduction of cane toads was not a good idea and that the toads would eat the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right.

3 Other Exponential Problems

3.1 Piles of Paper

A ream of paper (500 sheets) is about 50 mm thick, so that one sheet is about 0.1 mm thick. Take a sheet of paper, cut it in half and put the two halves one on top of the other. Cut this pile of 2 pieces in half and make a pile of 4 pieces. Keep cutting the pile in half and stacking the pieces up.

Now suppose you could make 42 cuts altogether (you'd need big scissors!). *How high would your final pile be?* Try making up a table like the one below to keep track of your pile. *Where would your pile reach to?* Kilometres might be a good unit to use eventually. *Write down a formula that tells you the height after n cuts. What units will you use?* Be careful!

Cut	**Height of pile**	
number	**in sheets**	**in mm**
1	2	0.2
2	4	0.4
3	8	0.8
4	16	1.6

This is an example of where maths lets you find an answer to something you can't actually do in real life.

3.2 Shoeing a Horse

A rich man sends his horse to the blacksmith to have 4 new horseshoes put on. Each shoe needs 5 nails. The blacksmith offers to charge either \$100 per nail (they're gold!), or 1c for the first nail, 2c for the second nail, 4c for the third nail, and so on, the cost doubling each nail. *Which offer should the rich man take?*

Think first which offer *you* would take. Then do some calculations. Don't forget to add up the total cost at the end. A calculator might be useful. *Did you pick the better offer? Was there much difference?*

Perhaps you might like to write down a function for the cost of the second offer after n nails and graph it. Write down a new version of the problem if each shoe needed 6 nails. What could the first offer be in this case?

3.3 Interest Rates

Once you have some money in the bank, you start to think about interest, and you might want to answer a question like the one below to work out how much money you will have some time in the future.

> *If the annual interest rate on a bank account is 12% compounded monthly and you deposit \$10, how much money will you have after 1 year? after 5 years? after 10 years?*

What does this mean? In simpler terms, it means that every month the bank will pay you an amount of interest equal to 1% (an annual interest rate of 12% means a monthly interest rate of 12%/12=1%) of the amount you have in the account at the end of the month. So, after the first month, the bank will pay into your account 1% of \$10 or $0.01\times\$10=\$0.10=$10c in interest, and you will then have

$$\$10 + \$0.10 = 1.01 \times \$10 = \$10.10$$

in your account.

After the second month, the interest will be 1% of \$10.10 or $0.01\times\$10.10=\$0.101=$10.1c, and you will then have

$$\$10.10 + \$0.101 = 1.01 \times \$10.10 = 1.01 \times (1.01 \times \$10) = \$10.201$$

in your account. Although the bank won't pay you the 0.1c, they leave it in for future calculations.

Can you see a pattern? At the start of each month, the new amount in your account will be the amount you had last month times 1.01.

Now can you answer the question above? Can you write down a formula using exponential notation for the amount in your account after n *months? How long before you have \$15? \$30?*

4 Solutions

Diagrams (and calculations) here were done on a TI-84CE graphics calculator.

Exponential Iteration

$1 \times 2 = 2 = 2^1$
$1 \times 2 \times 2 = 4 = 2^2$
$1 \times 2 \times 2 \times 2 = 8 = 2^3$
$1 \times 2 \times 2 \times 2 \times 2 = 16 = 2^4$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 = 32 = 2^5$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64 = 2^6$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128 = 2^7$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256 = 2^8$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512 = 2^9$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024 = 2^{10}$

Lots and Lots of Bacteria

You should end up with a table containing the numbers above and lots more. From your table, you can read off the answers to the questions. If you are using a calculator, it will probably switch to scientific notation when the numbers become large enough; powers of 10 are just like powers of 2, but much easier to write down. 10^3 is a 1 followed by three zeros, 10^{10} a 1 followed by 10 zeros, and so on.

Time	**Number of Bacteria**
after 1 minute	2
after 10 minutes	1024 $(=2^{10})$
after 20 minutes	$1,048,576$ $(=2^{20})$
after 1 hour	about 1.15×10^{18} $(=2^{60})$

The number of bacteria after n minutes is 2^n.

The Earth isn't covered with these bacteria because in reality the growth is not exponential but has a limit: environmental conditions ultimately limit growth and scientists continue to discover new antibiotics.

Malthus and Exponential Growth

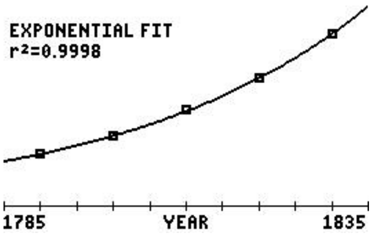
Thomas Malthus was worried about the world's population because he believed that population growth increased exponentially such as in the examples above, but that the food supply would only grow linearly (as a straight line — much more slowly). In other words, left unchecked, the human population would eventually exceed its food and other resources, leading to overcrowding, poverty, malnutrition, disease, crime and war. You can find out more about Malthus at *www.igc.org/desip/malthus*.

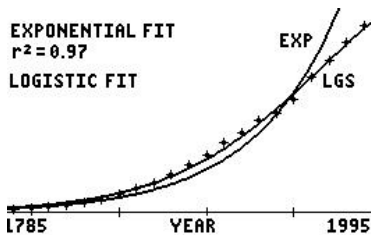
Starting at 1790, multiplying each number in the table by 1.35 gives a number close to the next number, up until 1860: the US population increased by about 35% every ten years from 1790 to 1860. The ten-year growth rate then decreased to values in the range 20–30%, and finally down to around $9\frac{1}{2}\%$ in the decade to 1990. A marked decline in growth occurred between 1910 and 1950, during the two world wars and the Great Depression. The growth rate picked up a bit after the wars (the baby boom: 1950–60), then slowly declined again.

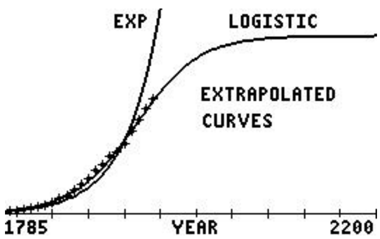
The left-hand figure below shows the exponential fit to the US population for the years 1790–1830. The fit is excellent (coefficient of determination r^2 very close to 1).

The middle figure shows exponential and logistic fits to the full data set 1785–1990. The exponential fit ($r^2 = 0.97$) is not quite as good now and doesn't follow the trend in later years. The logistic fit is much better.

The right-hand figure shows an extrapolation of the two curves to the year 2200. The exponential model predicts a US population in 2200 of more than 29 billion, the logistic model about 386 million, eventually stabilising at about 388 million.



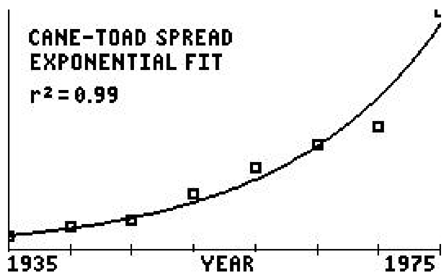




Cane Toads

The ratio of successive terms jumps about a bit, between 1.17 and 1.94, with a mean of 1.34. The population is therefore increasing roughly exponentially.

The exponential fit to the data looks reasonable (the value from 1969 is a bit low: a drought period from 1964 to 1969?). The curve of best fit is $y = 36.9 \times 1.08^t$ or, using *the* exponential function, $y = 36.9e^{0.0774t}$, where t is years since 1939.



Use a graphics calculator to find when $y = 1728$ and $y = 7619$ on the curve of best fit.[3] According to the exponential model, Queensland was overrun by cane toads between 1988 and 1989, Australia between 2007 and 2008. Clearly, and fortunately, there are some factors that restrict the spread of the cane toad. *Find out more?*

[3] Graph $y = 1728$ and $y = 7619$, and use *intersect*/*ISCT* to find in what year these lines intersect the curve of best fit.

Piles of Paper

1 sheet of paper = 0.1 mm thick.

After 1 cut, 2 (2^1) sheets of paper = 0.2 mm thick.
After 2 cuts, 4 (2^2) sheets of paper = 0.4 mm thick.
After 3 cuts, 8 (2^3) sheets of paper = 0.8 mm thick.
⋮
After 42 cuts, 4,398,046,511,104 (2^{42}) sheets of paper ≈ 439,804,651,110 mm thick, 439,805 km — a little more than the distance from the Earth to the Moon.

Shoeing a Horse

A horse needs 4 new horseshoes with 5 nails in each.

Nail	**Cost of nail**
1st	1c
2nd	2c
3rd	4c
4th	8c
5th	16c
6th	32c
7th	64c
8th	$1.28
9th	$2.56
10th	$5.12
⋮	⋮
19th	$2,621.44
20th	$5,242.88
Total	$10,485.75

If the man pays $100 per nail, it will cost him $2,000 to shoe the horse.

If he pays by the nail as in the table above (the total cost is the sum of all the costs in the second column), it will cost him $10,485.75, more than 5 times the flat rate. The choice is now obvious, and demonstrates clearly how rapidly exponential functions can increase.

Interest Rates

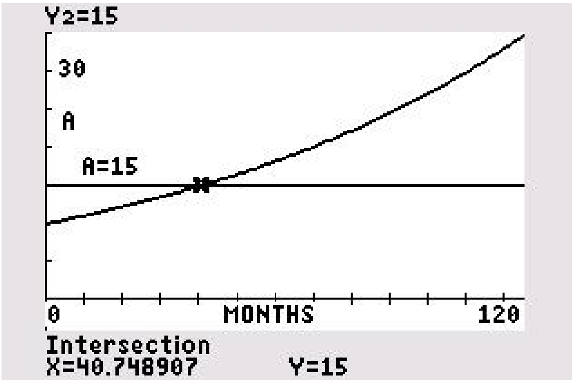
Deposit	Annual Interest Rate	Time Period	Value
\$10.00	12%	after 1 year	\$11.27 (=\$10.00 $\times$ 1.01^{12})
	compounded monthly	after 5 years	\$18.17 (=\$10.00 $\times$ 1.01^{60})
		after 10 years	\$33.00 (=\$10.00 $\times$ 1.01^{120})

A formula for the amount A in a bank account, given compound interest, is therefore

$$A(n) = D\left(1+\frac{i}{100}\right)^n,$$

where D is the initial deposit, $i\%$ the interest rate per month and n the number of months the money has been left in the bank.

It takes 41 months or 3 years 5 months before you have \$15, and 111 months or 9 years 3 months before you have \$30. Find these values by trial and error, by using the table feature on the graphics calculator or by graphing the function A with the number of months n as the independent variable, and finding where A equals 15 (figure below).



Amount in bank $A(n)$ versus month n, showing when the amount reaches \$15.

5 TI-84/CE Population-Modelling Programs

The TI-84 programs are available on the (free) CD *Graphics Calculator Activities* by Peter McIntyre and Margie Smith, or from Peter McIntyre at pdmcintyre@icloud.com. The TI-84CE programs are available from Peter McIntyre.

5.1 AUSPOP/AUSPOPCE: Australian population 1906–1996

These programs (used for the first figure in the Introduction) plot the population of Australia, fit and plot an exponential function and a logistic function, and plot a logistic curve from the Australian Bureau of Statistics (ABS). They also show extrapolation of the curves to predict the Australian population in the future.

Use: Run the program. The screen tells you which keys can be used at any given time. After the program has finished, you can [trace] the graphs on the screen or manually replot the data (Plot1) and any of the models (Y_1: exponential; Y_2: logistic; Y_3: ABS) by turning on/off the appropriate plot/functions in [y=]. You can also change the [window]. Press [stat] Edit to see the data and model values.

5.2 MALTHUS/MLTHUSCE: US population 1790–1990

These programs plot the data for 1790–1830 from the table of US population on page 5 and fit an exponential function of the form $P(t) = ab^t$, where a and b are constants. You can do this manually on the calculator — the programs just makes it easier. You might like to work out possible values of a and b manually using just the first two data points.

The programs also plot all the data from the table, and fit both an exponential function and a logistic function. They also show extrapolation of the curves to predict the US population in the future.

Use: Run the program and choose which plot you want. The screen tells you which keys can be used at any given time. After the program has finished, you can [trace] the graphs on the screen or manually replot the data (Plot1) and any of the models (Y_1: exponential; Y_2: logistic) by turning on/off the appropriate plot/functions in [y=]. You can also change the [window]. Press [stat] Edit to see the data and model values.

5.3 WORLDPOP/WLDPOPCE: World population 1940–2000

These programs (not used in the problems here) plot the world population, fit and plot an exponential function and a logistic function, and plot a logistic curve from the US Bureau of Statistics (USBS). They also show extrapolation of the curves to predict the world population in the future.

Use: Run the program. The screen tells you which keys can be used at any given time. After the program has finished, you can [trace] the graphs on the screen or manually replot the data (Plot1) and any of the models (Y_1: exponential; Y_2: logistic; Y_3: USBS) by turning on/off the appropriate plot/functions in [y=]. You can also change the [window]. Press [stat] Edit to see the data and model values.

After you have finished running any of these programs, delete lists FIT, EFIT, LFIT, POP, RESID and YR, and clear the functions Y_1–Y_3.