The geometry of stars

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Canberra Mathematical Association Conference
August 2018

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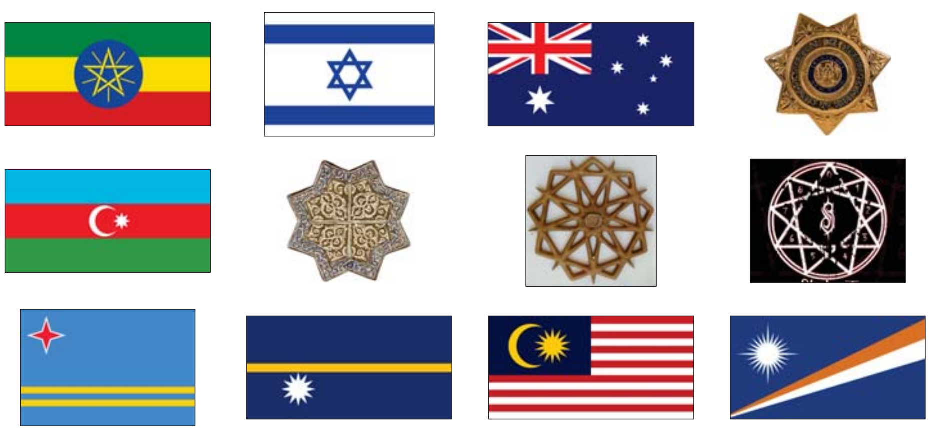
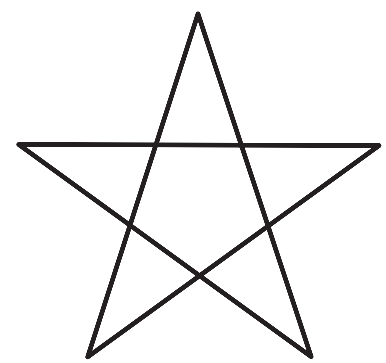
Outline

- Examples of geometrical star designs
- Angle sum of a triangle and other polygons
- Properties of a pentagram
- A method for constructing other stars
- Connections with number theory and complex numbers
- Angle sum of arbitrary stars
- Turtle programming with Python

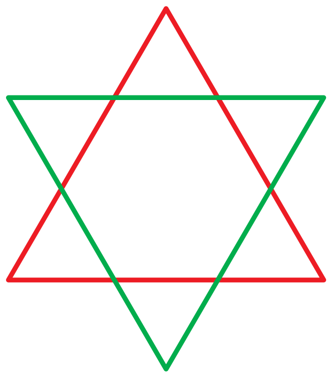
Examples – flags, badges, tiles & logos



Examples – 5 points

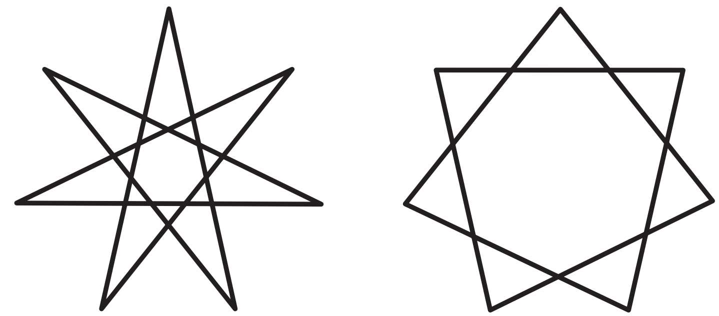


Examples – 6 points



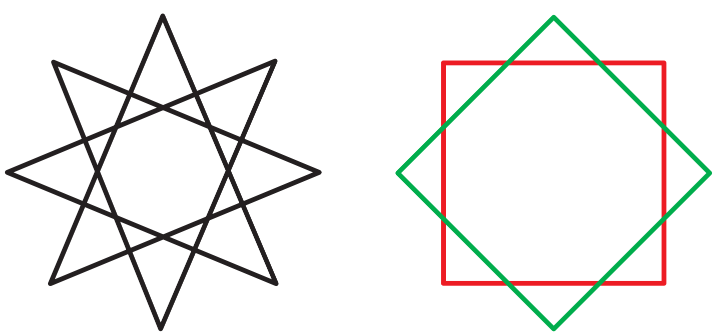
2 disconnected triangles

Examples – 7 points



2 different types

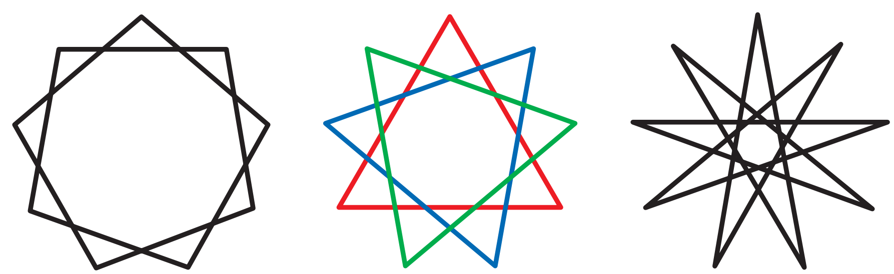
Examples – 8 points



connected disconnected

2 different types

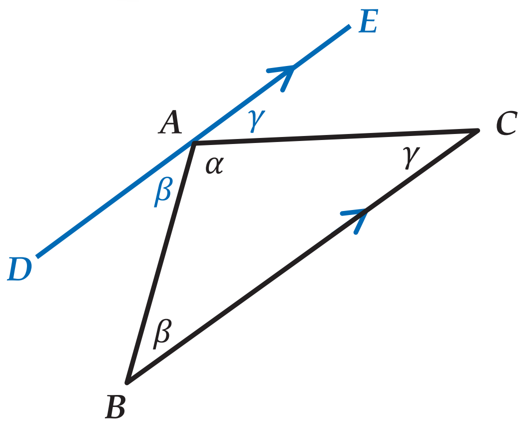
Examples – 9 points



connected disconnected connected

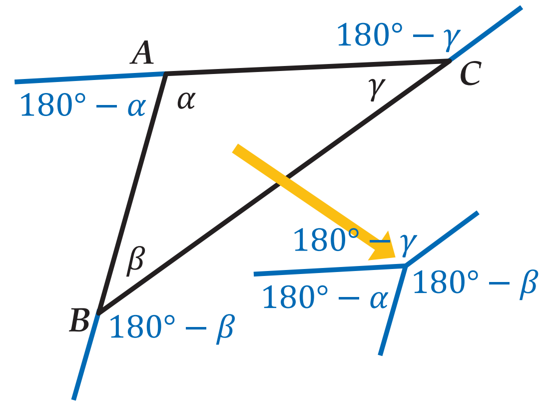
3 different types

Angle sum of a triangle – Method 1



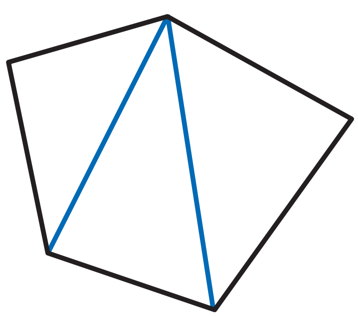
- Consider triangle ABC with angles α, β and γ
- Construct line-segment DE through A parallel to BC
- By alternate angles
  $\beta = \angle ABC = \angle DAB$
  $\gamma = \angle ACB = \angle EAC$
- By adjacent angles in a straight angle
  $\alpha + \beta + \gamma = 180°$

Angle sum of a triangle – Method 2



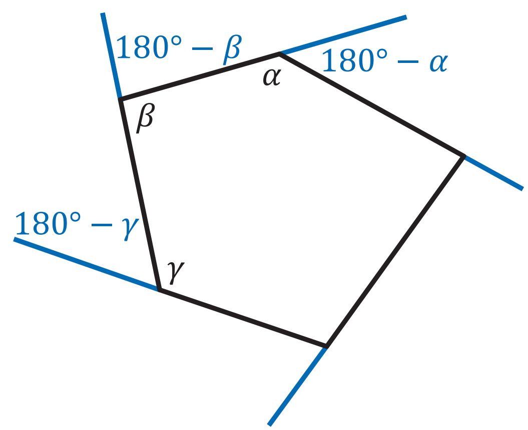
- Consider triangle ABC with angles α, β and γ
- Extend sides to form exterior angles $180° - \alpha$ etc.
- Shrink ABC to a single point with three exterior angles
- By adjacent angles in a revolution
  $(180° - \alpha) + (180° - \beta) + (180° - \gamma) = 360°$
- Hence
  $\alpha + \beta + \gamma = 180°$

Angle sum of a polygon – Method 1



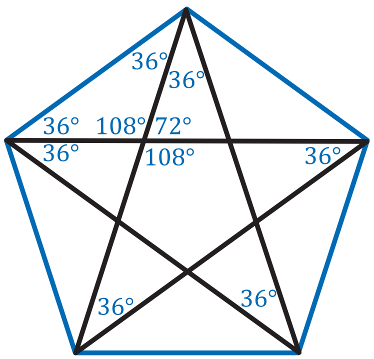
- Consider an n-sided polygon ($n = 5$ shown)
- Divide the polygon into $n - 2$ triangles by adding $n - 3$ diagonals
- By angle sum in each triangle, the total angle sum is $(n - 2) \times 180°$

Angle sum of a polygon – Method 2



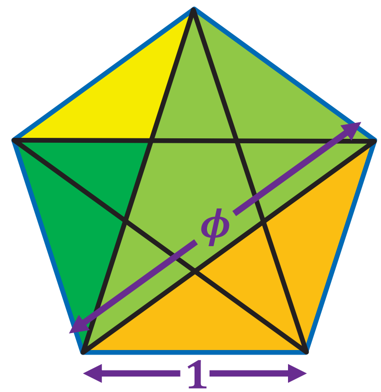
- Consider an n-sided polygon with angles α, β, γ,…
- Extend sides to form exterior angles $180° - \alpha$ etc.
- Shrink polygon to a single point with n exterior angles
- By adjacent angles in a revolution
  $(180° - \alpha) + (180° - \beta) + (180° - \gamma) + \cdots = 360°$
- Hence
  $\alpha + \beta + \gamma + \cdots = (n - 2) \times 180°$

Properties of a regular pentagram



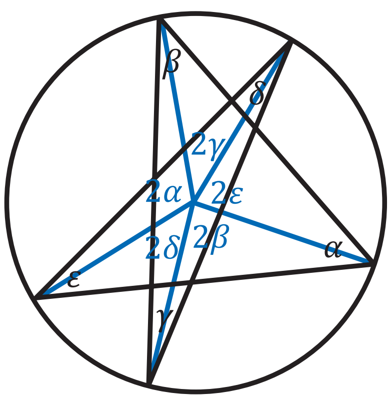
- Inscribe inside a regular pentagon
- Internal angles of pentagon are $3 \times 180° \div 5 = 108°$
- By isosceles triangles, angle at each vertex of regular pentagram is 36°, with sum 180°
- GeoGebra link: https://tinyurl.com/CMA2018pentagram

Properties of a regular pentagram



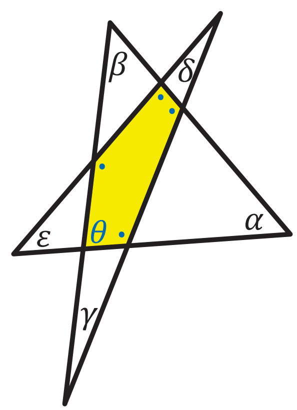
- Inscribe inside a regular pentagon
- Internal angles of pentagon are $3 \times 180° \div 5 = 108°$
- By isosceles triangles, angle at each vertex of regular pentagram is 36°, with sum 180°
- By similar triangles, the ratio of diagonal and side lengths is the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$, which is related to Fibonacci etc.

Angle sum of a cyclic pentagram



- Consider a cyclic pentagram with angles $\alpha, \beta, \gamma, \delta, \varepsilon$ at the vertices
- Construct radii through centre O opposite angle α
- By circle geometry theorem, the associated angle at O is 2α
- Similarly for all other angles
- By adjacent angles in a revolution $2\alpha + 2\beta + 2\gamma + 2\delta + 2\varepsilon = 360°$
- Hence $\alpha + \beta + \gamma + \delta + \varepsilon = 180°$

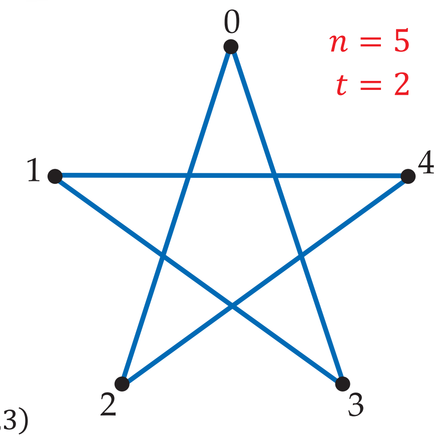
Angle sum of any pentagram



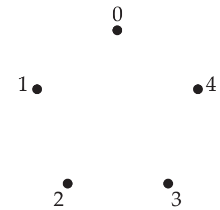
- Consider any pentagram with angles $\alpha, \beta, \gamma, \delta, \varepsilon$ at the vertices
- Consider triangle with angles α, β and third angle $\theta = 180° - \alpha - \beta$
- Similarly for the four other angles of the internal pentagon
- By angle sum of a pentagon $(180° - \alpha - \beta) + (180° - \beta - \gamma) + (180° - \gamma - \delta) + (180° - \delta - \varepsilon) + (180° - \varepsilon - \alpha) = 540°$
- Hence... $\alpha + \beta + \gamma + \delta + \varepsilon = 180°$

Constructing regular stars

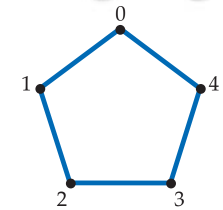
- Choose a number of vertices n
- Label n vertices around a circle $0, 1, 2, \dots, n-1$
- Choose a skipping number t
- For each $k = 0, 1, 2, \dots, n-1$, join vertex k to vertex $k + t \pmod{n}$
- Call this star $S(n,t)$
- The example shown is $S(5,2)$
- Reversing the order, it is also $S(5,3)$



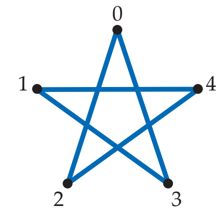
Constructing regular stars



$S(5,0) = S(5,5)$

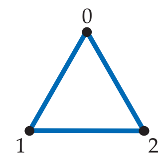


$S(5,1) = S(5,4)$

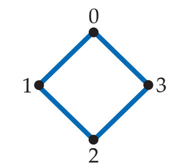


$S(5,2) = S(5,3)$

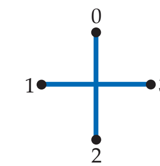
$S(2,1)$



$S(3,1)$

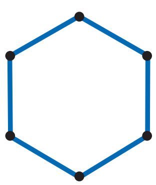


$S(4,1)$

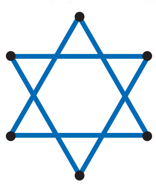


$S(4,2) = 2 \times S(2,1)$

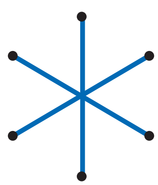
Constructing regular stars



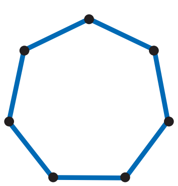
$S(6,1)$



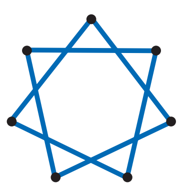
$S(6,2) = 2 \times S(3,1)$



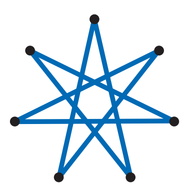
$S(6,3) = 3 \times S(2,1)$



$S(7,1)$

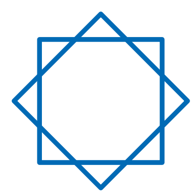


$S(7,2)$

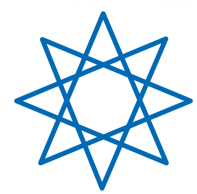


$S(7,3)$

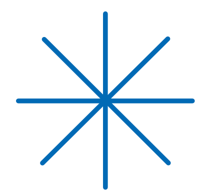
Constructing regular stars



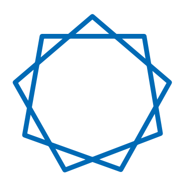
$S(8,2) = 2 \times S(4,1)$



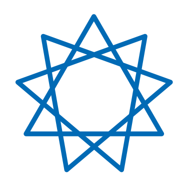
$S(8,3)$



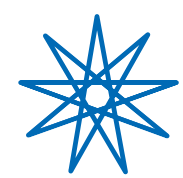
$S(8,4) = 4 \times S(2,1)$



$S(9,2)$



$S(9,3) = 3 \times S(3,1)$



$S(9,4)$

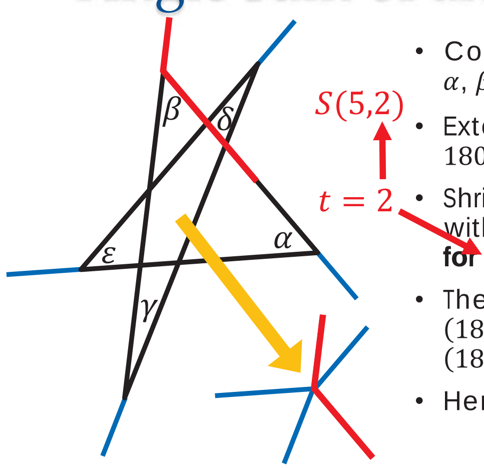
Stars and number theory

- $S(n,t) = S(n, n-t)$, so we can assume $0 \le t \le \frac{n}{2}$
- $S(n,1)$ is an n-sided polygon
- If $d = \gcd(n,t)$, then $S(n,t)$ is d rotated copies of $S\left(\frac{n}{d}, \frac{t}{d}\right)$
- Therefore $S(n,t)$ is connected if and only if $\gcd(n,t) = 1$
- For $n \ge 3$, the number of different connected stars with n vertices is $\frac{\varphi(n)}{2}$, where φ is the Euler-totient function

Stars and complex numbers

- It is well known that if $\omega = \operatorname{cis}\left(\frac{2\pi}{n}\right)$ is the principal n^{th} root of unity, the powers $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ form a regular n-gon in the complex plane
- In general, if $z = \omega^t$ is another n^{th} root of unity, then the powers $1, z, z^2, z^3, \ldots, z^{n-1}$ form $S(n,t)$
- This follows from index laws and the fact that $z^n = 1$ so indices can be reduced modulo n
- Adjusting the magnitude of z introduces spiralling

Angle sum of any pentagram



- Consider any pentagram with angles $\alpha, \beta, \gamma, \delta, \varepsilon$ at the vertices
- Extend sides to form exterior angles $180° - \alpha$ etc.
- Shrink pentagram to a single point with n exterior angles **which account for 2 full revolutions**
- Then $(180° - \alpha) + (180° - \beta) + (180° - \gamma) + (180° - \delta) + (180° - \varepsilon) = 2 \times 360°$
- Hence
  $$\alpha + \beta + \gamma + \delta + \varepsilon = 180°$$

Angle sum of any star

- Consider the star $S(n,t)$ with angles $\alpha_1, \alpha_1, \ldots, \alpha_n$ at the vertices (note that we do not need to assume they are equal!)
- The exterior angle at vertex k is $180° - \alpha_k$
- Since t vertices are skipped at each step, it takes t revolutions to trace around the full star
- Hence the sum of exterior angles is

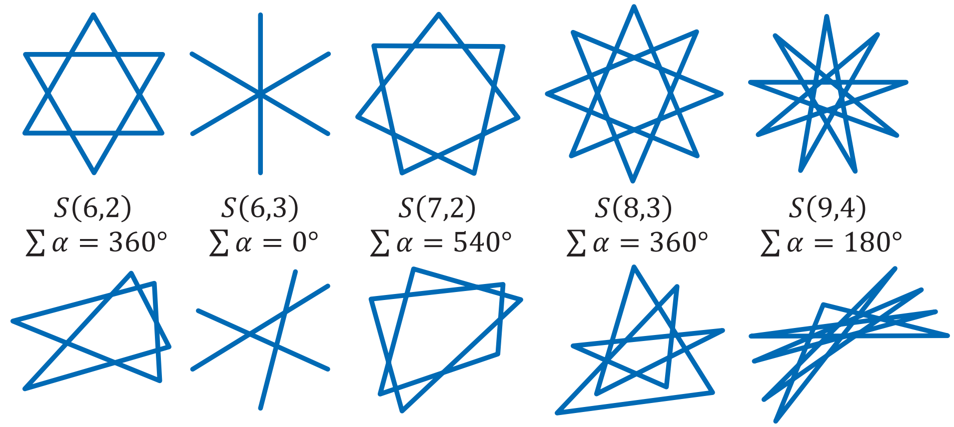
$$2t \times 180° = t \times 360° = \sum_{k=1}^{n} (180° - \alpha_k) = n \times 180° - \sum_{k=1}^{n} \alpha_k$$

- Therefore the sum of angles at the vertices is $(n - 2t) \times 180°$

Angle sum = $(n - 2t) \times 180°$

- **Theorem:** If $1 \leq t \leq \frac{n}{2}$, the angle sum of the, possibly irregular, star $S(n,t)$ is $(n-2t) \times 180°$.
- **Corollary 1:** Letting $t = 1$, the angle sum of an n-sided polygon is $(n-2) \times 180°$.
- **Corollary 2:** If $1 \leq t \leq \frac{n}{2}$, the regular star $S(n,t)$ has angles of $\left(1 - \frac{2t}{n}\right) \times 180°$ at each vertex.

Angle sum = $(n - 2t) \times 180°$

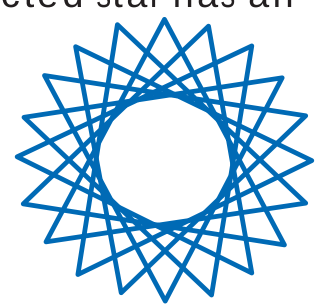


Angle sum = $(n - 2t) \times 180°$

- Example: Which regular connected star has an angle of 54° at each vertex?
- Solution: $\left(1 - \frac{2t}{n}\right) \times 180° = 54°$

$$1 - \frac{2t}{n} = \frac{54}{180} = \frac{3}{10}$$

$$\frac{2t}{n} = \frac{7}{10} \quad \text{so} \quad \frac{t}{n} = \frac{7}{20}$$



$\gcd(t,n) = 1$, since the star is connected, hence $n = 20$ and $t = 7$, so the star is $S(20,7)$

Turtle programming with Python

- Free online Python compiler: https://trinket.io/library/trinkets/create?lang=python
- Example: Plot regular $S(20,7)$, with exterior angles of $180° - 54° = 126°$

```
from turtle import *
for k in range(20):
    forward(100)
    left(126)
```

must keep indents here to ensure these two commands are part of the 'for loop'

Turtle programming with Python

- Example: Plot any regular connected $S(n,t)$, $\gcd(n,t) = 1$

```
from turtle import *
n=20       # number of vertices
t=7        # skipping number
s=100      # side length
e=360*t/n  # exterior angle at vertices
for k in range(n):
    forward(s)
    left(e)
```

- For $\gcd(n,t) \neq 1$ see https://tinyurl.com/CMA2018stars

Other Turtle features

```
right(…)        # turn right by given angle
back(…)         # move backwards by given distance
dot(…)          # draw dot of given size
penup()         # lift pen to move without drawing
pendown()       # resume drawing
pensize(…)      # any decimal value greater than 0, default is 1
pencolor(…)     # e.g. 'red', 'blue' etc., including quotes
begin_fill()    # turn on colour filling, must be paired with…
end_fill()      # turn off colour filling
hideturtle()    # hides arrowhead while tracing
speed(…)        # 1 = slowest, 100 = fastest
tracer(0,0)     # turns off tracing, plots final output only
update()        # must put this at the end if using tracer(0,0)
```

References

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Image sources

- Aruba flag: https://en.wikipedia.org/wiki/Flag_of_Aruba
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