

Chris Wetherell

St Francis Xavier College, ANU Extension &
Australian Mathematics Trust

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cjtwetherell@gmail.com

Domino challenges

- Can all 28 dominos be arranged end-to-end in a closed circuit?
- If all dominos with at least one 6 are removed, can the rest be arranged into a circuit?
- Can the 28 dominos be split into two groups of 14, each of which can be arranged into a circuit?

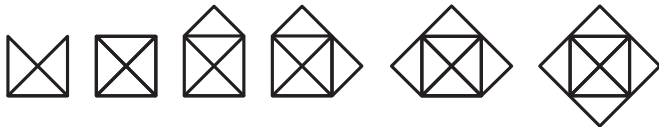
About MCYA

- The Mathematics Challenge for Young Australians (MCYA) is run by the Australian Mathematics Trust (AMT)
- There are two stages:
 - Challenge stage (4 weeks in March – June)
 - Enrichment stage (16 weeks in June – September)
- Both stages are open to all students in Years 3-10
- Schools are free to choose how and when to run it
- The MCYA is a small part of the AMT's Olympiad training program
- More info at <https://www.amt.edu.au/mathematics/mcya/>

	Stage	Middle Primary Yr 3-4	Upper Primary Yr 5-6	Junior Yr 7-8	Intermediate Yr 9-10	Senior Yr 11-12
Open invitation	MCYA Challenge stage	✓	✓	✓	✓	
	MCYA Enrichment stage	✓	✓	✓	✓	
	Australian Mathematics Competition (AMC)	✓	✓	✓	✓	✓
School invitation	Australian Intermediate Mathematical Olympiad (AIMO)		✓	✓	✓	
State director invitation	Australian Mathematical Olympiad Committee (AMOC) Senior Contest		✓	✓	✓	✓
	Australian Mathematical Olympiad (AMO)			✓	✓	✓
AMOC invitation	School of Excellence			✓	✓	✓
	IMO Selection School			✓	✓	✓
	International Mathematical Olympiad (IMO)				✓	✓

Traceable diagrams

- Which of the following diagrams can you trace
 - WITHOUT lifting your pen/pencil, and
 - WITHOUT retracing any lines?



- Is there a general rule for when it is possible to trace a diagram and when it isn't?
- Test your ideas with other diagrams of your own. Ask another student to try to draw them.

Odd degrees

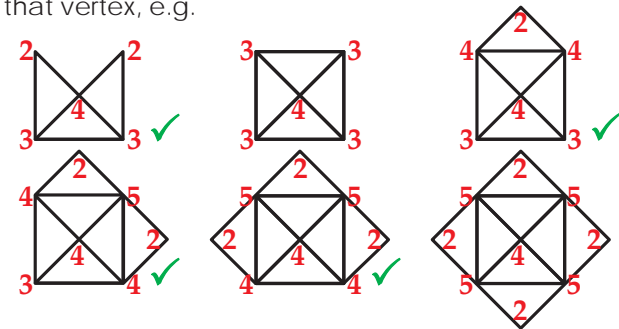
- Conjecture*:**
 - A diagram cannot be traced if it has more than two vertices with odd degree.
 - If a diagram has exactly two vertices with odd degree, the tracing must start at one of them and finish at the other.

* fancy name for a 'guess'

- Test this conjecture with other diagrams of your own.
- Can you explain why it works?
- What if the vertices all have even degree? Can the diagram be traced? If so, where can the tracing start and finish? Explain.

Degree of a vertex

- A *vertex* is just another name for a corner.
- The *degree* of a vertex is the number of lines meeting at that vertex, e.g.



- Redraw the diagrams you think are possible, but pay careful attention to where you need to start and finish

Euler trails and circuits

A diagram can be traced if, and only if, either:

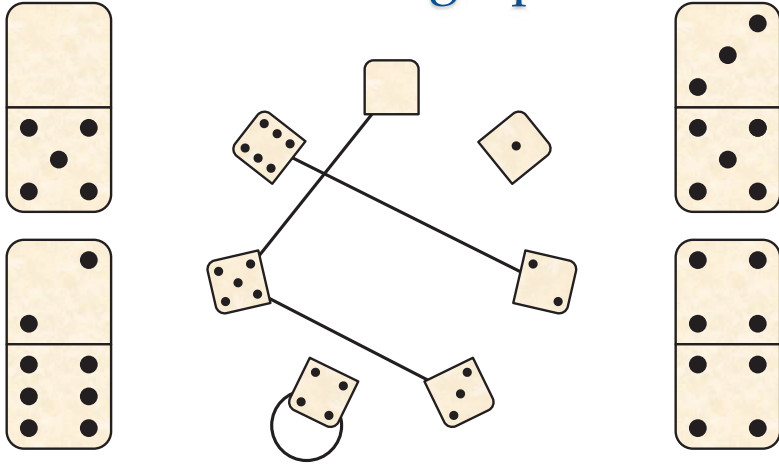
- there are exactly two vertices with odd degree (the tracing must start at one and finish at the other)

or

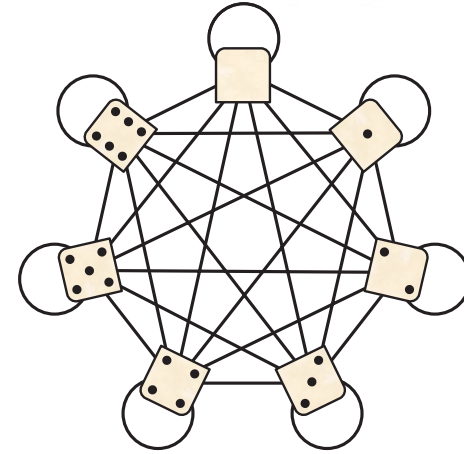
- all vertices have even degree (the tracing can start at any vertex, but it must finish at the same vertex, so it forms a circuit)

In **graph theory** these concepts are known as an *Euler trail* and an *Euler circuit*, respectively.

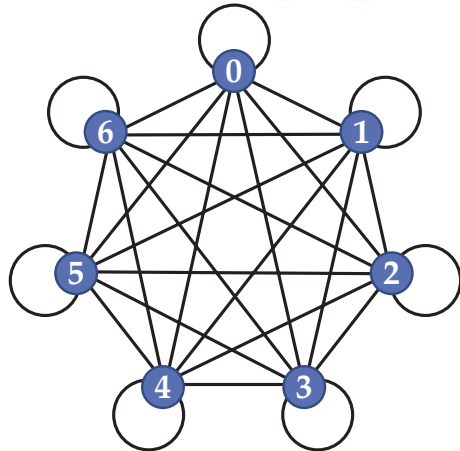
Domino graph



Domino graph

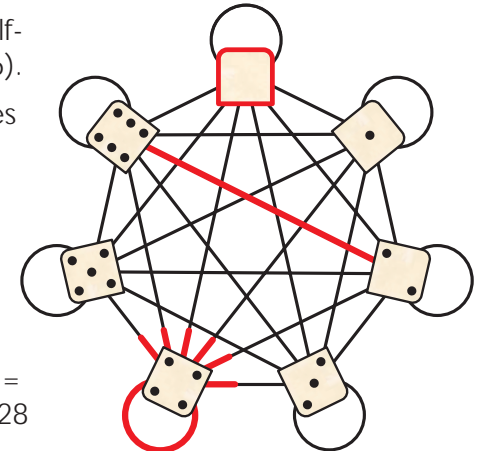


Domino graph

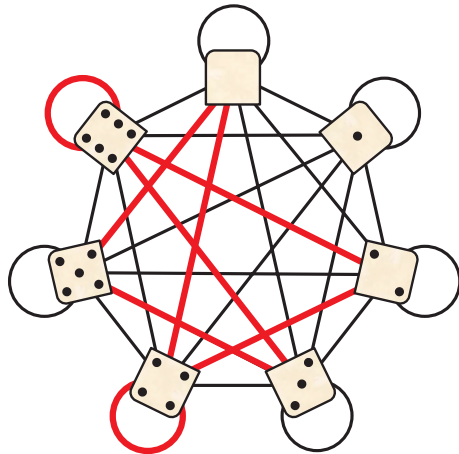
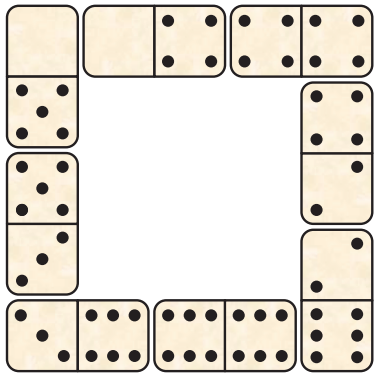


Vertices, edges and degree

- Each vertex represents a half-domino (blank, 1, 2, 3, 4, 5, 6).
- Each edge between vertices represents a unique domino
- 'Doubles' are loops
- Degree of each vertex = 8
- Total number of edge-ends = $7 \times 8 = 56$
- So total number of dominos = number of edges = $56 \div 2 = 28$

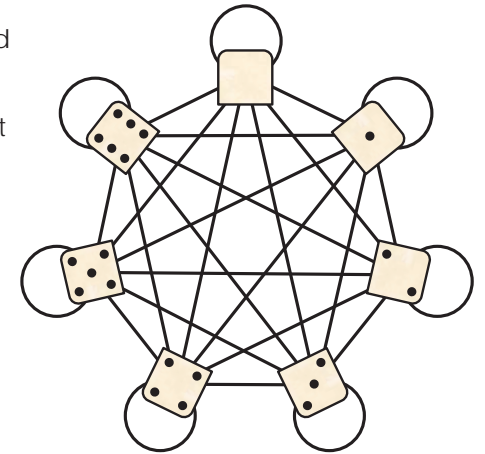


Chains and circuits of dominos



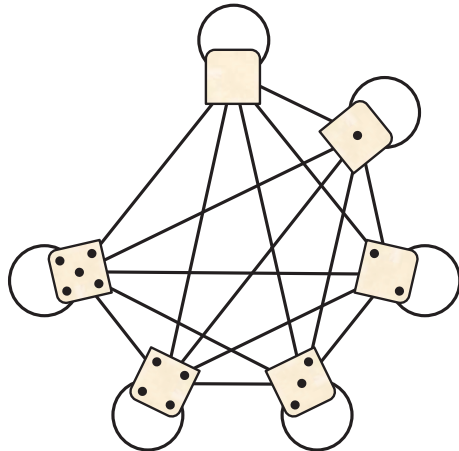
Circuits with all or some dominos?

- Q:** Can all 28 dominos be arranged end-to-end in a circuit?
- A:** Yes! Each vertex has degree 8 (even) so there is an Euler circuit



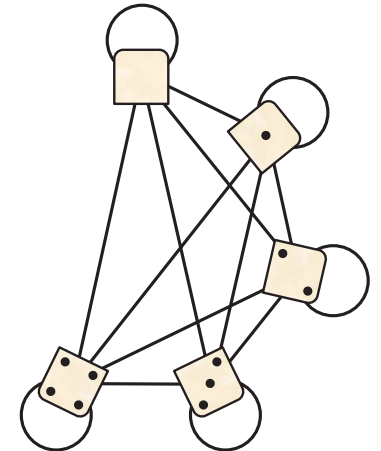
Circuits with all or some dominos?

- Q:** Can all 28 dominos be arranged end-to-end in a circuit?
- A:** Yes! Each vertex has degree 8 (even) so there is an Euler circuit
- Q:** If all dominos with at least one 6 are removed, can the rest be arranged into a circuit?
- A:** No! Each remaining vertex now has degree 7 (odd)



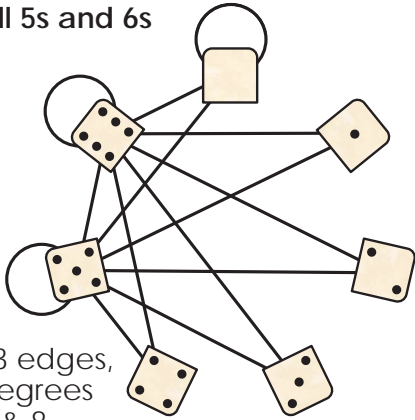
Circuits with all or some dominos?

- Q:** Can all 28 dominos be arranged end-to-end in a circuit?
- A:** Yes! Each vertex has degree 8 (even) so there is an Euler circuit
- Q:** If all dominos with at least one 6 are removed, can the rest be arranged into a circuit?
- A:** No! Each remaining vertex now has degree 7 (odd)
- Q:** What if we also remove all the 5s?
- A:** Yes! Each vertex has degree 6 (even)



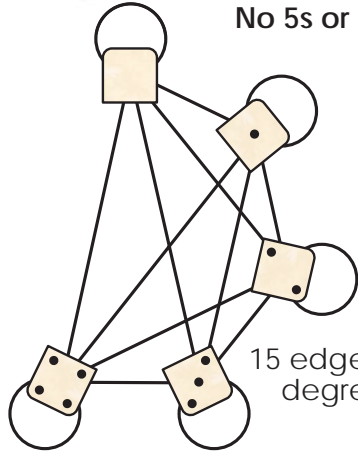
Circuits with two groups of 14?

All 5s and 6s



13 edges,
degrees
2 & 8

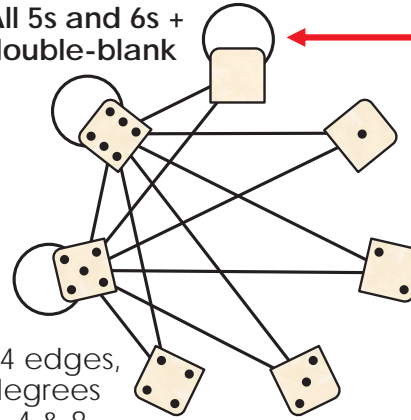
No 5s or 6s



15 edges,
degree
6

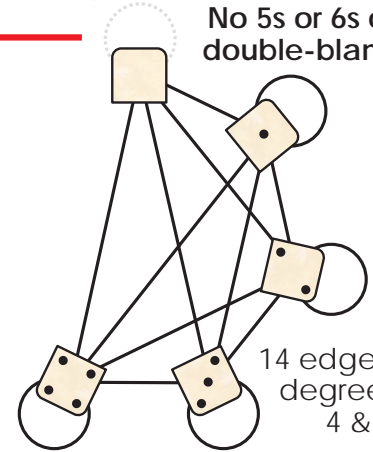
Circuits with two groups of 14!

All 5s and 6s +
double-blank



14 edges,
degrees
2, 4 & 8

No 5s or 6s or
double-blank



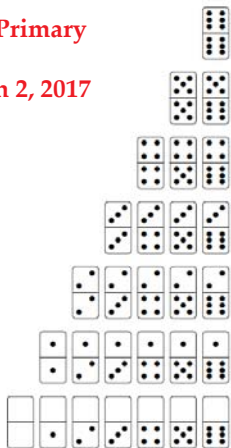
14 edges,
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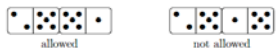
MP2 Domino Chains

Dominoes are rectangular tiles, twice as long as they are wide, with two sets of dots separated by a line as shown below. The number of dots in each set varies from 0 to 6. There are 28 different dominoes.

Middle Primary
Year 3-4
Question 2, 2017



In a game of dominoes, they are placed end-to-end to form a chain, always matching the number of dots where the dominoes join. For example, the [2,5] and [1,5] dominoes can be placed as in the first diagram, but not the second.



a Make a chain of nine dominoes that includes all the dominoes that have at least one set of six dots.

Each domino has a *domino product*, that is, the product of the number of dots at one end with the number of dots at the other. For example, the domino product of [2,5] is $2 \times 5 = 10$.

b How many dominoes have an odd domino product? Justify your answer.

c List the ten dominoes which have the largest domino products.

d Make a chain of nine dominoes so that the sum of their domino products is 196.

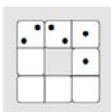
Excerpt from Student Problems Booklet

Marking Scheme

- a A correct chain: 1 mark
- b Correct answer (6) with justification: 1 mark
- c Correct list of 10 dominoes: 1 mark
- d A correct chain: 1 mark

Solutions to Extensions

1. The maximum number of dots on a domino is 12 and there is only one domino with 11 dots. So, if none of Taj's dominoes has 12 dots, then the total number of dots can't be more than $11 + 10 + 10 = 31$, which is less than 33. So one of Taj's dominoes has 12 dots. If neither of the other dominoes has 11 dots, then the total number of dots can't be more than $12 + 10 + 10 = 32$, which is also less than 33. So one of Taj's dominoes has 11 dots. To get a total of 33 dots the last domino must have 10 dots. The only dominoes with 10 dots are [6,4] and [5,5]. So the two collections are [6,6], [6,5], [6,4] and [6,6], [6,5], [5,5].
2. In a domino loop, each set of dots is matched by the set of dots on the domino next to it. So the sets of dots in a domino loop occur in identical pairs. Hence the total number of dots in a domino loop is even. Since 17 is odd, John is wrong.
3. Since two dominoes cannot be joined along their long edge, a loop must have at least 4 dominoes.
A loop cannot have the domino [0,0] more than once. So there are at least three dominoes with at least 1 dot in each. A loop cannot have the domino [0,1] more than once. So there are at least two dominoes with at least two dots each. Hence the total number of dots is at least 5. From Extension 2, the total number of dots in a domino loop is even. Hence the total number of dots is at least 6. Here is a loop of 4 dominoes with a total of 6 dots.



So the smallest number of dots in a loop of dominoes is 6.

4. The following table shows the number of dots on each domino in a 10-set.

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1		2	3	4	5	6	7	8	9	10	11
2			4	5	6	7	8	9	10	11	12
3				6	7	8	9	10	11	12	13
4					8	9	10	11	12	13	14
5						10	11	12	13	14	15
6							12	13	14	15	16
7								14	15	16	17
8									16	17	18
9										18	19
10											20
Total	0	3	9	18	30	45	63	84	108	135	165

So the total number of dominoes in a 10-set is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 66$.

The total number of dots is $0 + 3 + 9 + 18 + 30 + 45 + 63 + 84 + 108 + 135 + 165 = 660$.

Alternatively, notice that the average number of dots per domino amongst the 21 dominoes in the top row and last column of the table above is 10. The average is also 10 for the remaining 17 dominoes in the second row and second last column, and for the remaining 13 dominoes in the third row and third last column, and so on. So the total number of dots is $10 \times (21 + 17 + 13 + 9 + 5 + 1) = 10 \times 66 = 660$.

Alternatively, notice that among the 66 dominoes in a 10-set there are exactly 12 domino halves that have 0 dots, exactly 12 domino halves that have 1 dot, exactly 12 domino halves that have 2 dots, and so on. So the total number of dots is $12 \times (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 12 \times 55 = 660$.

Excerpt from Teacher Guide

Solutions

- a There are seven dominoes with six dots at either end. The following chain of nine dominoes contains all seven: [0,6] [6,6] [6,5] [5,4] [4,6] [6,3] [3,2] [2,6] [6,1].
There are other valid chains. □
- b The product of an even number with any other number is even. The product of two odd numbers is odd. Hence the only dominoes with odd domino products are [1,1] [1,3] [1,5] [3,3] [3,5] [5,5]. Thus there are exactly six dominoes with an odd domino product. □
- c The following table shows the domino product for each domino.

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1		1	2	3	4	5	6
2			4	6	8	10	12
3				9	12	15	18
4					16	20	24
5						25	30
6							36

The 10 largest products, including repetitions, are 36, 30, 25, 24, 20, 18, 16, 15, 12, 12.

Hence the required dominoes are [6,6] [5,6] [5,5] [4,6] [4,5] [3,6] [4,4] [3,5] [3,4] [2,6]. □

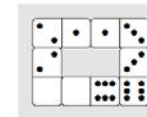
- d Adding various domino products from the table in Part c, we find the nine largest domino products 36, 30, 25, 24, 20, 18, 16, 15, 12 sum to 196. These domino products come from dominoes [6,6] [5,6] [5,5] [4,6] [4,5] [3,6] [4,4] [3,5], along with either [3,4] or [2,6].
Choosing [2,6], we get the following chain: [2,6] [6,6] [6,5] [5,5] [5,4] [4,4] [4,6] [6,3] [3,5].
There are other acceptable chains. □

Discussion

1. This problem is a modification of one proposed by Lorraine Motterhead.
2. It involves selecting then arranging dominoes into chains.
3. It also involves forming products of small integers.

Extensions

1. Taj picked up 3 dominoes and noticed that they had a total of 33 dots. List the two possible collections of 3 dominoes.
2. A *domino loop* is a chain with its two ends joined. Here is an example.



Excerpt from Teacher Guide

John made a domino loop, added all the dots and said the total was 17. Explain why John was wrong.

3. Find a domino loop that has the smallest total number of dots.
4. Dominoes come in different sets. The set described above is called a *6-set* (because 6 is the largest number of dots in any domino half), but you can also have 7-sets and 8-sets for example. How many dominoes are there in a 10-set? How many dots would there be altogether in this set?

References

Australian Mathematics Trust. 2017. *Mathematics Challenge for Young Australians*.

<https://www.amt.edu.au/mathematics/mcya/>

McAvaney, K. et al. 2017. *Mathematics Challenge for Young Australians, Challenge Stage Teacher Guide*.

Wikipedia. 2018. *Eulerian path*.

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Image source

Dominoes animated gif

<https://giphy.com/gifs/loop-domino-hrkMkmWuX6EWk>