

All teachers are teachers of English

Chris Wetherell

Radford College & Australian Mathematics Trust

chris.wetherell@radford.act.edu.au

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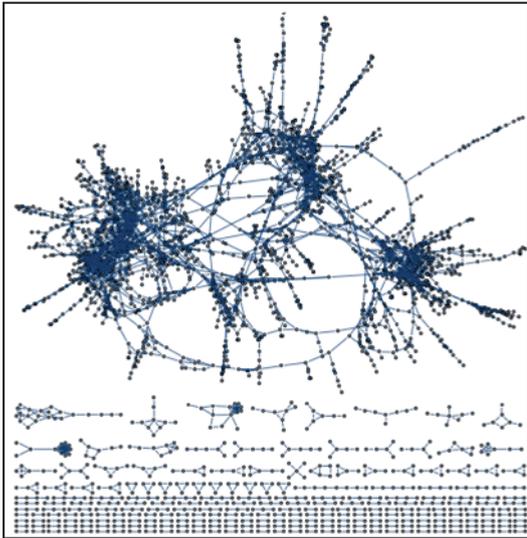
Doublets

- Puzzle attributed to Lewis Carroll who published them in Vanity Fair from March 1879
- Also commonly known as 'word ladders'
- Given two words of the same length, the goal is to turn one into the other by changing 1 letter at a time
- E.g. SHIP→DOCK ?
SHIP→SHOP→SHOT→SOOT→SOOK→SOCK→DOCK
- E.g. APE→MAN ?
APE→APT→OPT→OAT→MAT→MAN
- Solver at <http://ceptimus.co.uk/wordladder.php> gives 2 solutions: shortest and common words only

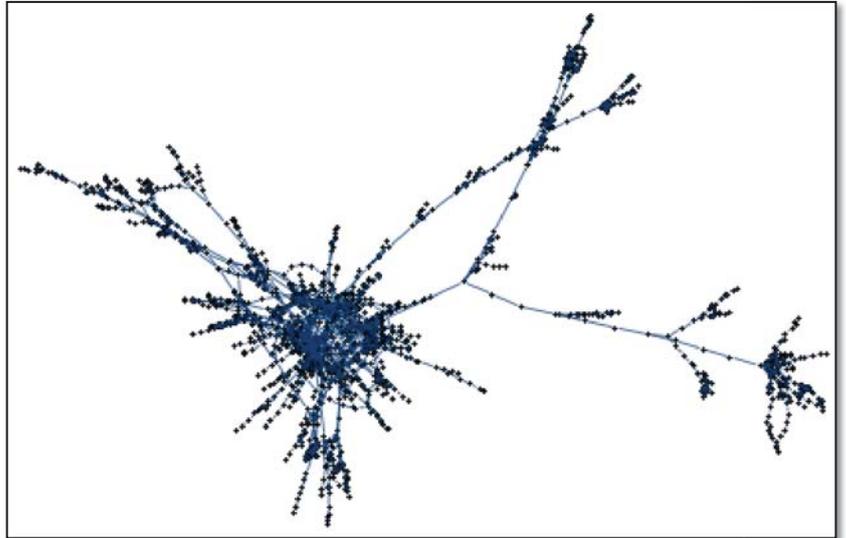
An undirected graph

- The doublet puzzle defines a simple undirected graph (or network) where:
 - vertices (nodes) are words of a given length
 - two words are joined by an edge if their letters differ in exactly one position
- For a given pair of words, there exists a solution to the puzzle if and only if there is a *path* between the corresponding vertices of the doublet graph
- The 'best' solution is the *shortest path* between the vertices
- These concepts are somewhat subjective since they depend on the dictionary being used

Graphs in *Mathematica*



5-letter doublet graph
(excluding aloof words)

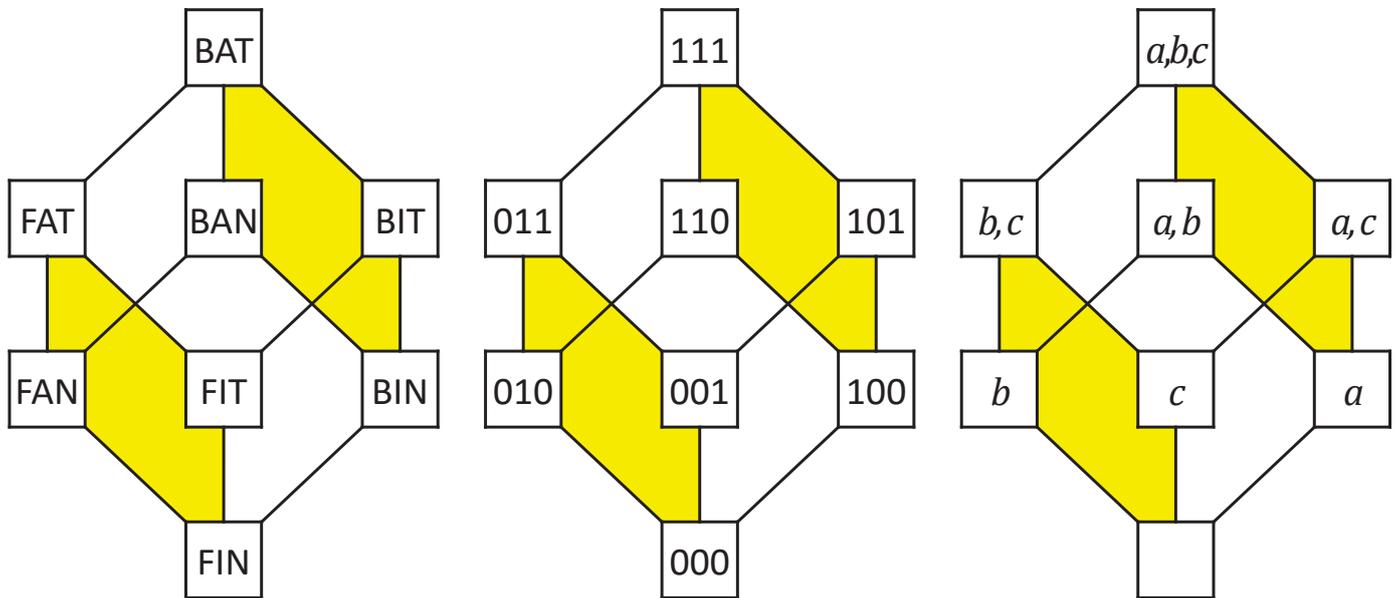


Largest connected component
of the 6-letter doublet graph

Graph theory concepts

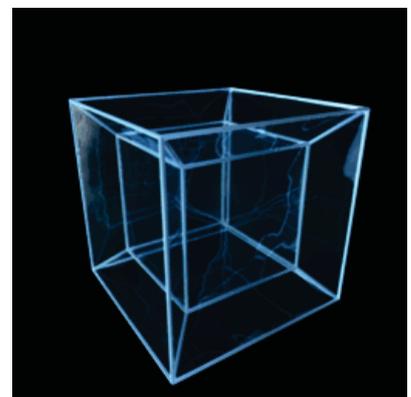
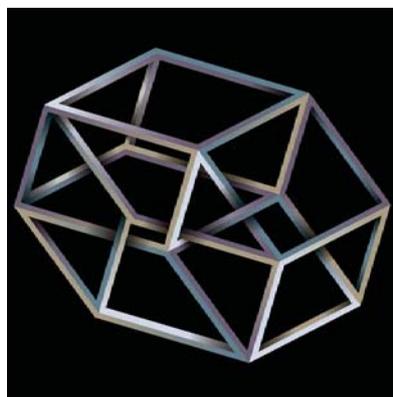
Doublet	Graph
Puzzle solution	Path between vertices
Best solution	Shortest path between vertices
Collection of all mutually solvable words	Connected component
'Aloof' words (coined by Donald Knuth)	Isolated vertices, e.g. LYNX
Length of longest puzzle	Diameter of the largest connected component, e.g. CHARGE → COMEDO
Collection of n words all differing in the same position	Complete graph K_n , e.g. <div style="text-align: center;"> </div>

Coordinates and subsets

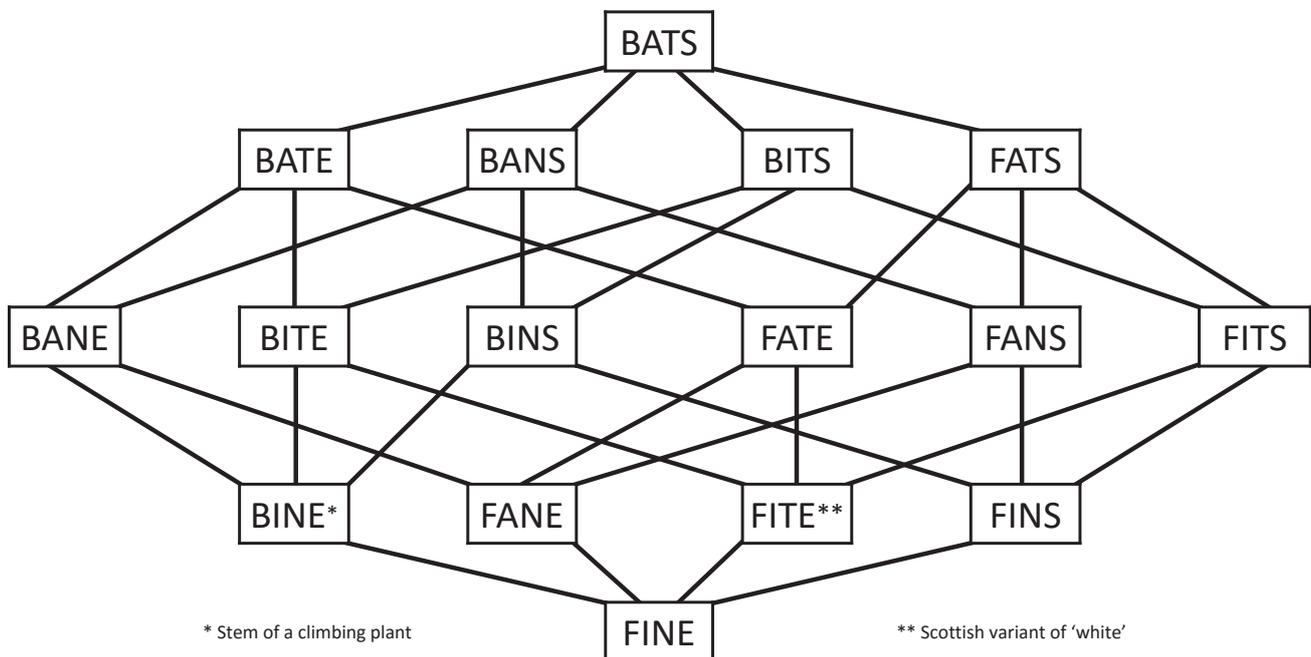


Tesseract

(4-dimensional hypercube)



Tesseract



Induced subgraphs

- Since we can find large(ish) values of n for which a complete graph K_n exists, many familiar graphs can be found as a subgraph of a doublet graph
- An *induced* subgraph is a collection of vertices and *all* of the edges that join them in the full graph
- E.g. A chain of length 2 can be formed in lots of ways:

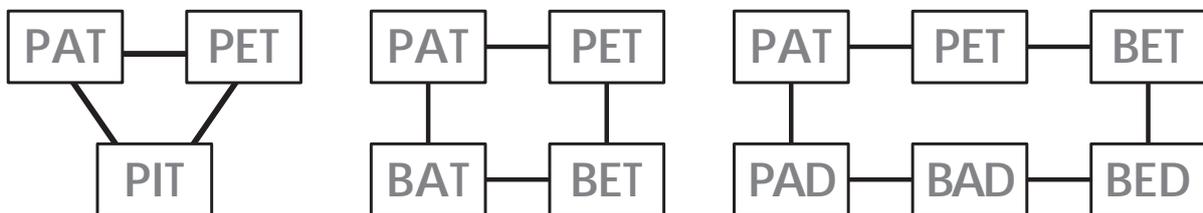
PAT→**PET**→**BET** or **PAT**→**PET**→**PIT**

The first example is an induced subgraph, but the second is not because adding an extra edge from PIT back to PAT would form a cycle of length 3

- Which graphs can be found as *induced* subgraphs?

Chains and cycles

- Assuming that no redundant steps are included, any solution to a doublet puzzle forms an induced subgraph isomorphic to a chain, e.g. length 6:
SHIP→**SHOP**→**SHOT**→**SOOT**→**SOOK**→**SOCK**→**DOCK**
- Cycles of length 1 and 2 are impossible, since the graph is simple, but cycles of length 3, 4 and 6 are possible, e.g.

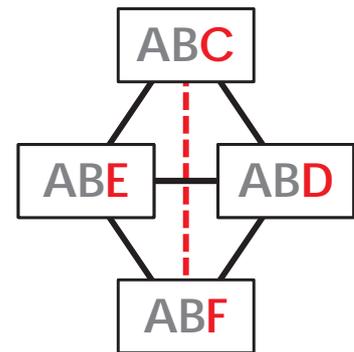
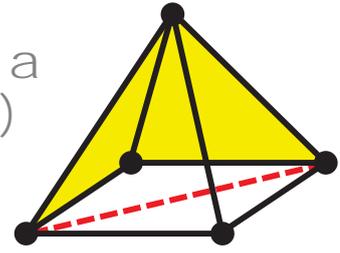


Induced cycle of length 5?

- Suppose the first 'word' is **ABC**
- If letters change in only 1 position, then this forms the complete graph K_5 , e.g. **ABC**, **ABD**, **ABE**, **ABF**, **ABG**
- If letters in 3 (or more) positions are changed, then at least six steps are required to arrive back at **ABC**
- Therefore letters must change in exactly 2 positions
- Since there are 5 steps, but only 2 positions change, the same position must change twice in a row, e.g. **ABC**→**ABD**→**ABE**
- But an extra edge can be added from **ABE** to **ABC** across the 'middle' of the cycle of length 5, hence such a cycle cannot form an induced subgraph

Induced pyramids?

- Suppose we wish to find an induced subgraph isomorphic to the graph of a square-based pyramid (wheel graph)
- Any triangular face (cycle of length 3) must be formed by three words all differing in the same position
- Any other triangular face which shares a common edge must have its third vertex also differing in the same position
- But this creates an extra edge
- The same argument rules out any pyramid with larger base, but also shows that the tetrahedron is possible (K_4)



Further exercises

7. Find other examples for the cube and tesseract.
8. Find eight words forming a cycle of length 8 which is an induced subgraph.
9. Is it possible for a cycle of length 6 to be induced if only two letter positions are changed throughout?
10. Can a cycle of length 7 be an induced subgraph?
11. A variation of the doublet puzzle is to allow a new type of move where the letters are rearranged but not replaced. That is, two words can also be joined by an edge if they are anagrams of each other. E.g. **APE** → **APT** → **TAP** → **MAP** → **MAN** is 1 step shorter. Can a cycle of length 5 or square-based pyramid form an induced subgraph for this modified puzzle?

References

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Exercises – hints

Note: some solutions depend on the dictionary in use and obscurity of words

1. Use the solver at <http://ceptimus.co.uk/wordladder.php>
Hovering over the first or last word in a list shows you all words connected to it, which could be used to check additional answers for questions 2 and 3
2. **3 letters:** jar, eggs, antelope, possessive pronoun, recede, philosophy;
4 letters: unattractive, unholy, rebounding sound, church song, flightless bird, alright, one time, bone, vacuum, egg, jars, water, axe-like tool, attractive, symbol of life; **5 letters:** planet, sea, under, sweetener, titter, pie, position (x3)
3. **3 letters:** tea & Greek letter; **4 letters:** quartz & antelope; **5 letters:** nectar(y)
4. E.g. SENT, ...
5. E.g. HAT, ... (shouldn't require any proper nouns or abbreviations)
7. E.g. PUT → something you put things in; PET → domesticator of pets
8. Use a subgraph of a subgraph previously discussed
9. Yes, now find an example
10. Yes, now find an example
11. Cycle of length 5 can now be induced; now find an example
For pyramids, adapt the argument for the regular puzzle