MARBLE TOWERS

An AMT Lesson Card

An activity suitable for Australian years 3–12

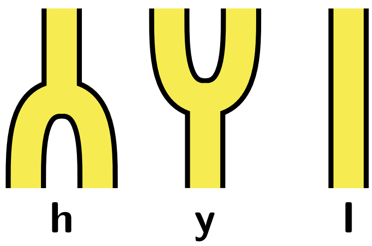
AUSTRALIAN MATHS TRUST

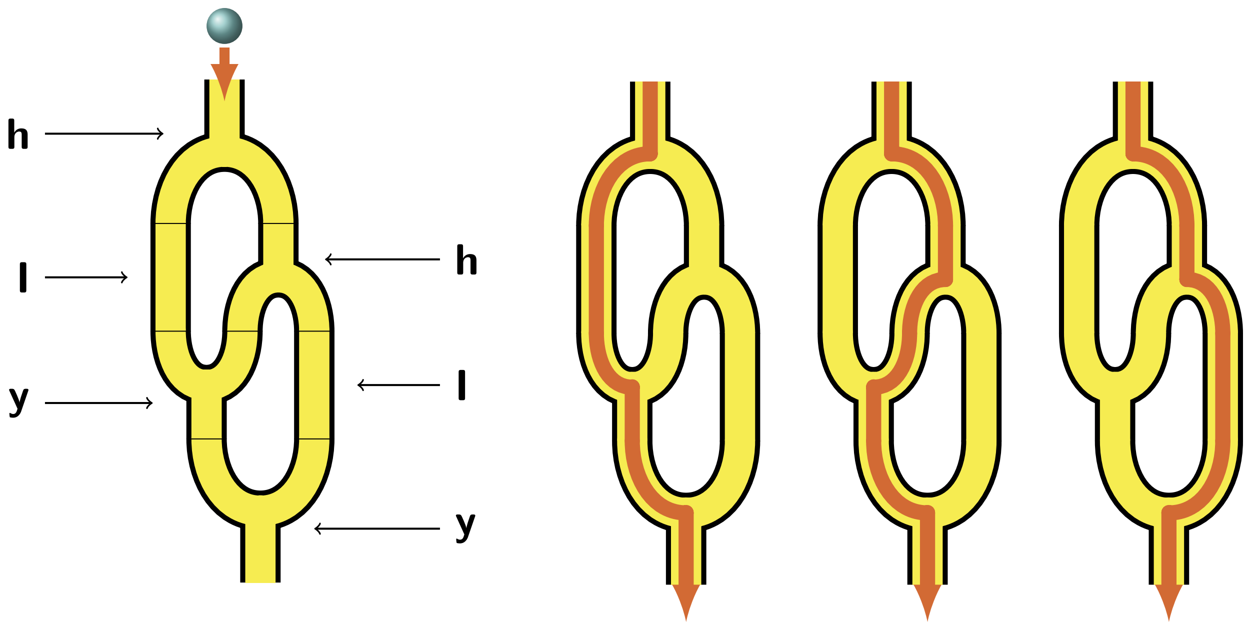
Learning areas: algorithmic thinking, spatial and logical reasoning, systematic counting, number patterns, networks, binary numbers. Links to the applicable Australian Curriculum content descriptors are on page 6.

Resources: Paper. Visit www.amt.edu.au/resources-for-the-classroom for links to the Australian Curriculum content descriptors, full solutions and additional resources for this and other activities.

Marble Towers

You are building a marble tower with three types of tubing called **h**-tubes, **y**-tubes and **l**-tubes, as shown on the right. All three types of tubing come in different sizes so they can be slotted together in various ways. A design for a marble tower with two **h**-tubes, two **y**-tubes and two **l**-tubes is shown on the left below. As in this example, marble towers have one opening at the top and one at the bottom.

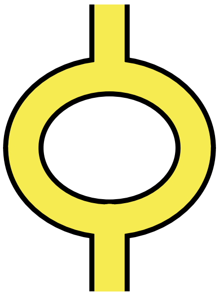


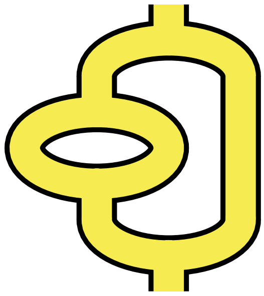


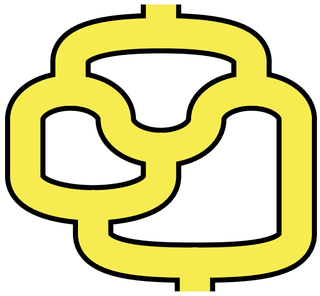
When a marble dropped in the top opening hits the junction of an **h**-tube, it randomly falls to the left or right. How many different paths could the marble take through the entire tower from top to bottom? One way to find out is to carefully list all the possible paths. In this example there are three.

Challenges

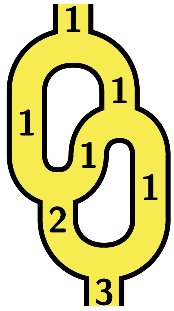
(a) For each of the following marble towers, list all possible paths.

i. 

ii. 

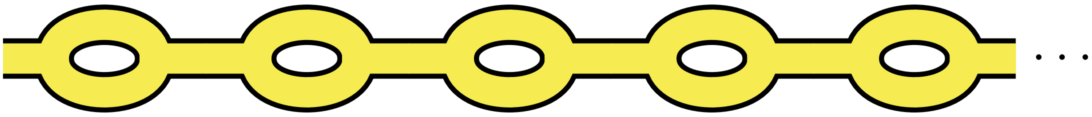
iii. 

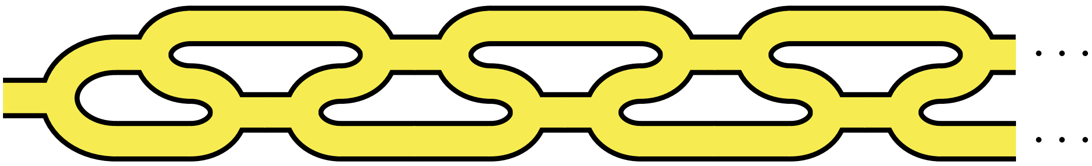
Here is a quicker way to find the number of paths without listing them all. Starting at the top and working down, label each tube with the number of ways that the marble can get there: wherever an **h**-tube splits into two branches, copy the number above into both branches, and wherever the two branches of a **y**-tube join, label the tube below with the sum of the two numbers above.



(b) Use this shortcut to check the total number of paths you found in (a).

(c) Which special numbers occur in the path labels for the following marble towers? (Note that the diagrams are sideways.)

i. 

ii. 

(d) Using the towers in (c) as a starting point, design two marble towers which each have 8 paths in total. Design two more towers with 8 paths which look different to your first two solutions.

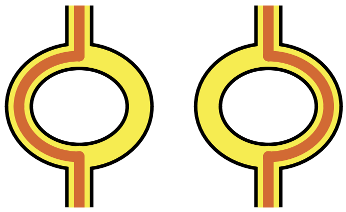
(e) For any numbers up to 10 not already accounted for by the examples in (a) or (c), design as many towers as you can with that number of paths.

(f) Investigate which whole numbers equal the number of paths for at least one tower and, when possible, how to build the shortest such tower.

(g) If multiple openings at the bottom are allowed, investigate how to build a tower whose paths represent any row of Pascal's triangle.

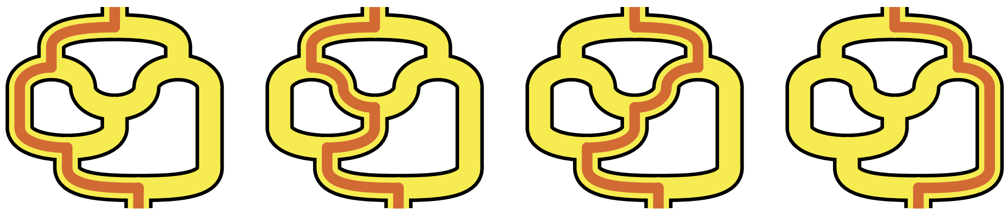
Solutions

(a) i.

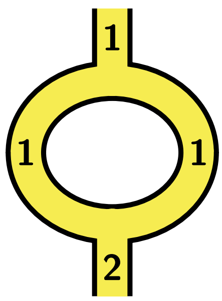


ii.

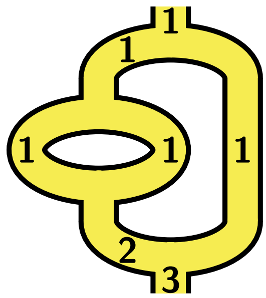
iii.



(b) i.



ii.



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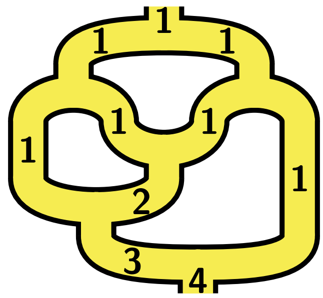
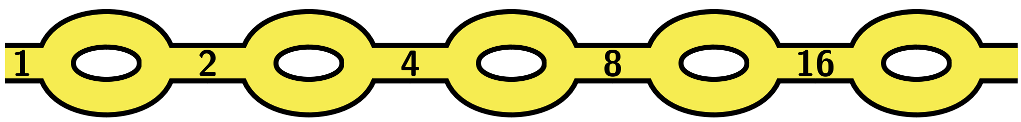


(c) It is possible to count the paths without labelling all sections of tubing. Wherever two branches of a **y**-tube join, follow the path on the left upwards until the next number is found, even if this means going all the way up to the 1 at the top opening. Similarly, find the next number following the path upwards on the right. Label the base of the **y**-tube with the sum of these two numbers. This shortcut has been used in the following solutions, in particular to highlight the special numbers that arise. Remember that these diagrams are sideways.

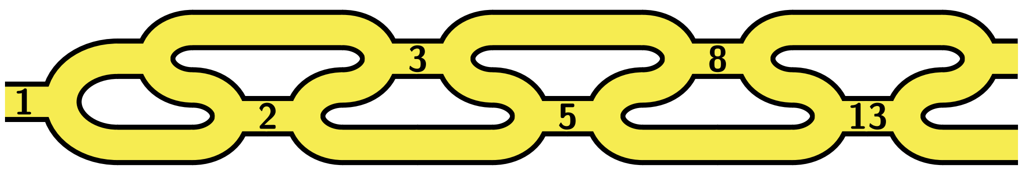
i.



. . .

Powers of 2: 1, 2, 4, 8, 16, . . .

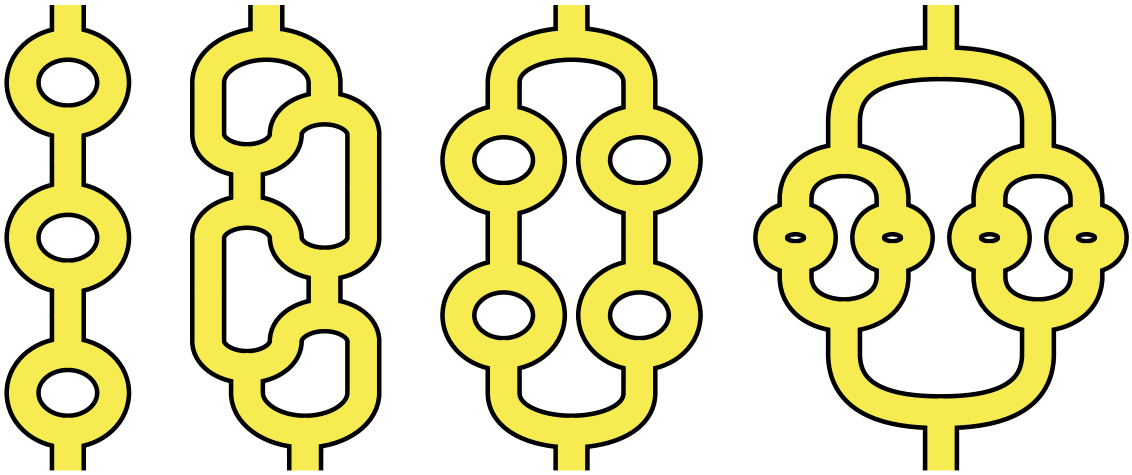
ii.



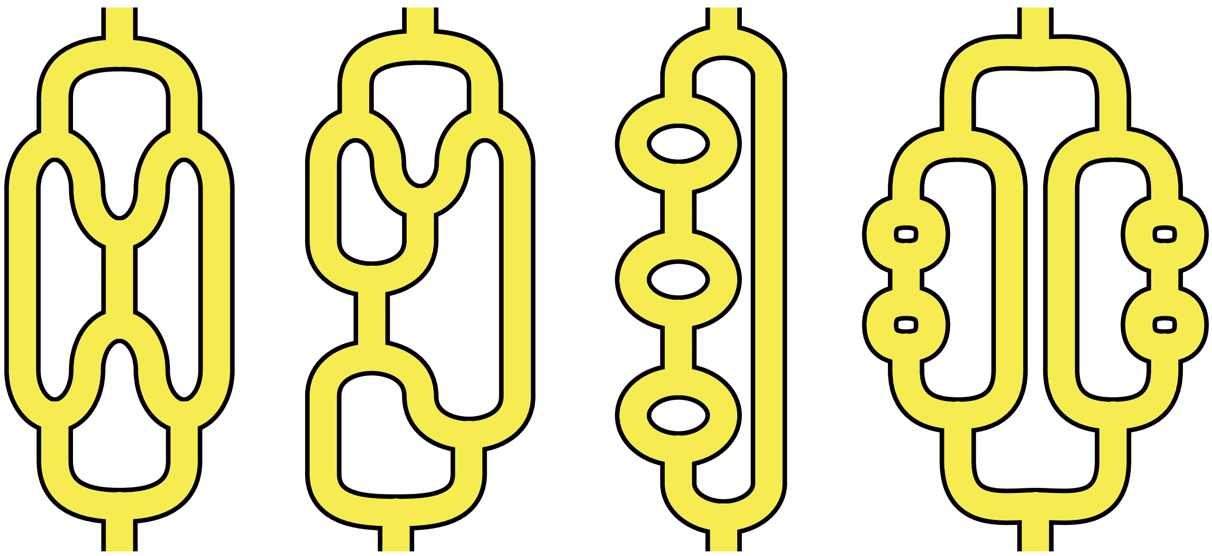
. . .
. . .

Fibonacci numbers: (1,) 1, 2, 3, 5, 8, 13, . . .

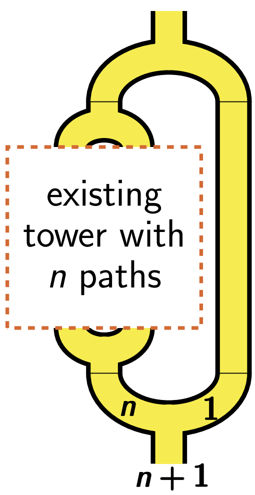
(d) The first two of these towers are based on (c) i. and ii. There are many other solutions for the remaining two designs.



(e) A tower with one path can be made from a single **l**-tube. Parts (a) and (c) account for towers with 2, 3, 4, 5 or 8 paths. Here are examples of towers with 6, 7, 9 and 10 paths, respectively. There are many others.

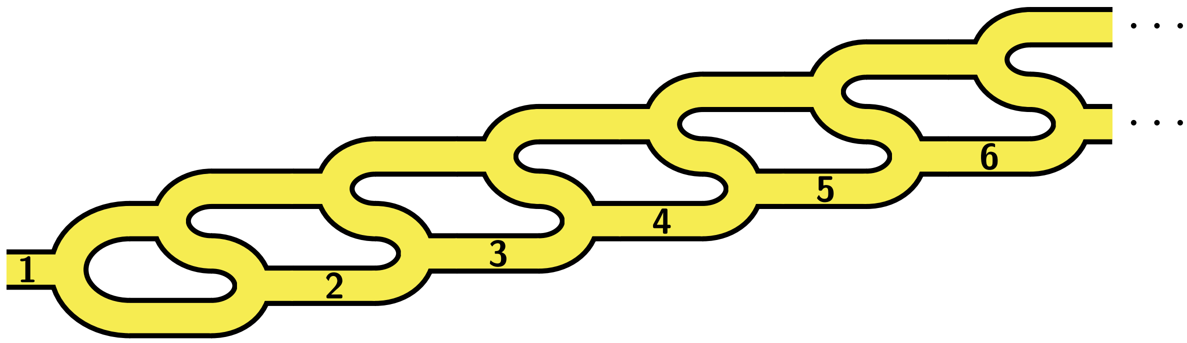


(f) Every whole number is the number of paths for at least one marble tower. Given any existing tower, we can always increase the number of paths by exactly one by connecting one more **h**-tube at the top, one more **y**-tube at the bottom, and one more **l**-tube between them, as illustrated.

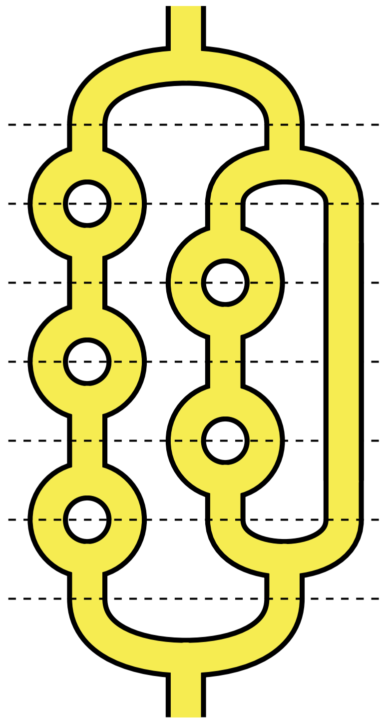


Starting with a single **l**-tube with one path, we can apply this trick as often as we like to make a tower with any number of paths.

There are other ways to construct a tower with any number of paths. For example, by adapting the Fibonacci tower in (c) ii., we have the following design which can be completed after any number of sections by removing the last **h**-tube. (Note that the diagram is sideways.)



Finally, we observed in (c) i. that any power of 2 can be achieved by stringing together a number of **h**-**y** pairs. Moreover, any whole number can be uniquely represented as a sum of distinct powers of 2; this is equivalent to its *binary* representation, where we use only the digits 0 and 1. For example, 13 can be written as $8+4+1$, or 1101 in binary, and so a tower for 13 can be built by exploiting the known towers for 8, 4 and 1, as shown.

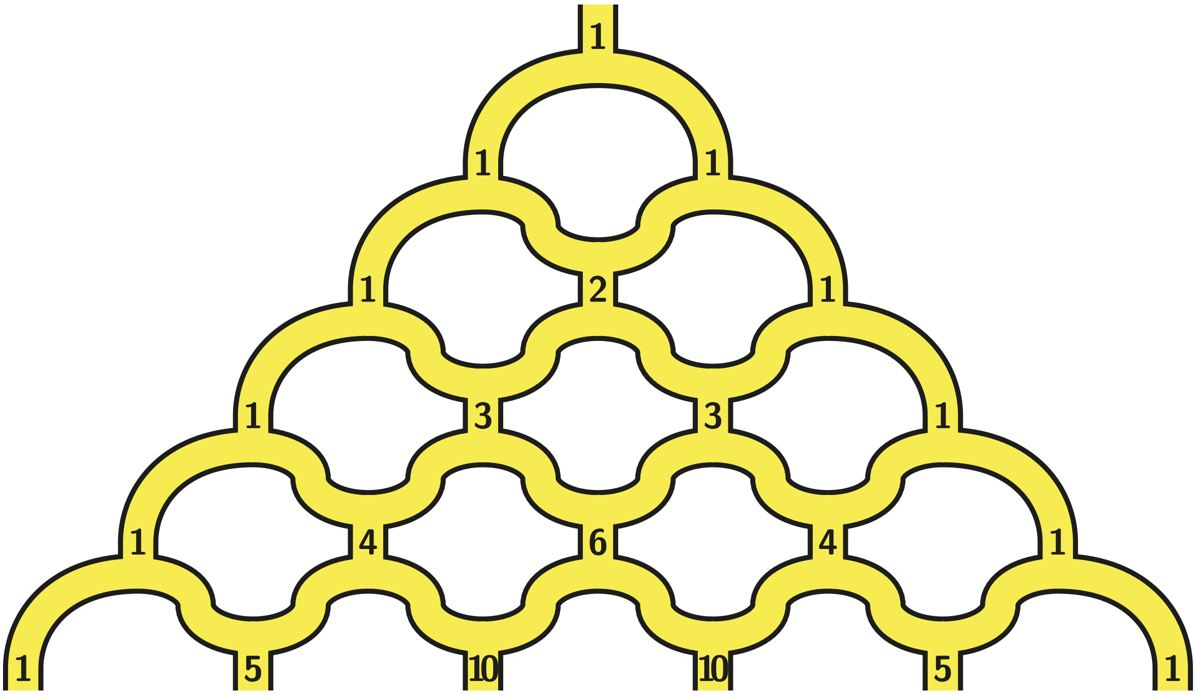


This tower requires eight levels of tubing, as indicated by the dashed lines. Other solutions for 13, including the Fibonacci tower and the other general strategies described above, will require more levels than this. When using this binary trick to build a tower with n paths, the number of levels needed is twice the number of binary digits of n (or 2 less than this if n happens to be a power of 2, since we can remove the **h**-tube at the top and **y**-tube at the bottom).

(g) Pascal's triangle is an infinite triangular arrangement of numbers with 1s down the sides and the property that every other term is the sum of the two above.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal's triangle has important applications in a number of areas of mathematics, such as algebra and probability. The marble tower shown here is the equivalent of the famous *Galton Board*, where balls randomly bounce to the left or right as they fall through a triangular array of pegs.



For further hints and tips, contact mail@amt.edu.au.

Acknowledgment

This activity was inspired by *Arc Routes*, Problem 5 on the Intermediate paper of the 2013 Computational and Algorithmic Thinking (CAT) competition: https://www.amt.edu.au/cat-competition.

Australian Curriculum content descriptors

The following is not intended to be an exhaustive list, but indicates how the above acitivity aligns with various stages of the mathematics curriculum.

- Year 3-4, ACTDIP010 Define simple problems, and describe and follow a sequence of steps and decisions (algorithms) needed to solve them
- Year 4, ACMNA081 Explore and describe number patterns resulting from performing multiplication
- Year 5-6, ACTDIP019 Design, modify and follow simple algorithms involving sequences of steps, branching, and iteration (repetition)
- Year 6, ACMNA133 Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence
- Year 7-8, ACTDIP029 Design algorithms represented diagrammatically and in English, and trace algorithms to predict output for a given input and to identify errors
- Year 9-10, ACTDIP040 Design algorithms represented diagrammatically and in structured English and validate algorithms and programs through tracing and test cases