Integer partitions and symmetric poylnomials

Concepts and activities for the senior classroom

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Algebraic combinatorics

- Wikipedia definition: "Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra, notably group theory and representation theory, in various combinatorial contexts and, conversely, applies combinatorial techniques to problems in algebra."
- Key topics relevant to senior curriculum:
  - Integer partitions
  - Young tableaux
  - Symmetric functions

Integer partitions: definition

A **partition** of an integer n is a way of writing n has a sum of positive integers. For example, the five partitions of 4 are:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

More formally, a partition is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ (possibly infinite) of non-negative integers such that

$$\lambda_1 \geq \lambda_2 \geq \cdots ,$$

where for some r, $\lambda_{r+1} = \lambda_{r+2} = \cdots = 0$. We do not distinguish between two partitions which differ only by a string of zeroes at the tail. For example,

$$(2,1) = (2,1,0) = (2,1,0,0) = (2,1,\underbrace{0,\ldots,0}_{m \text{ zeroes}})$$

Integer partitions: some language

- The non-zero λ_i are called **parts**.
- The number of parts is the **length** of a partition, denoted $l(\lambda)$.
- The sum of the parts is called the **weight** of a partition, and is denoted by $|\lambda|$.
- If n is the weight of a partition λ, we say that λ is a partition of n, and write $\lambda \vdash n$.
- The number of times an integer i occurs in a partition is called the **multiplicity** of i in λ and is denoted by m_i.

How many partitions of n?

- Let $p(n)$ denote the number of partitions of n.
- Bad news: no closed formula for $p(n)$.
- Good news: there is a generating function and it presents an opportunity for students to explore/learn two things:
  - We can use something algebraic to encode information in a non-obvious way.
  - A new way to think about expanding products of multinomials.

The generating function

- The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty}(1 + x^i + x^{2i} + x^{3i} + \cdots).$$

- The index of each term in the bracket $(1 + x^i + x^{2i} + x^{3i} + \cdots)$ encodes the number of times the integer i occurs in the partition.
- To find $p(n)$ only finitely many terms are needed in each bracket, and only finitely many brackets need to be multiplied.
- To find a specific n, we choose one term from each bracket in order to produce an x^n term, and count the number of ways this can be done.

An example calculation of $p(n)$

To calculate $p(5)$ we need to expand the product

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5)$$

We have the following choices:

$$\begin{aligned}
x^5\times 1\times 1\times 1\times 1 &\implies (1,1,1,1,1)\\
x^3\times x^2\times 1\times 1\times 1 &\implies (2,1,1,1)\\
x^2\times 1\times x^3\times 1\times 1 &\implies (3,1,1)\\
x\times x^4\times 1\times 1\times 1 &\implies (2,2,1)\\
x\times 1\times 1\times x^4\times 1 &\implies (4,1)\\
1\times x^2\times x^3\times 1\times 1 &\implies (3,2)\\
1\times 1\times 1\times 1\times x^5 &\implies (5)
\end{aligned}$$

Permutations

- Denote by $[n]$ the set $\{1, 2, \ldots, n\}$.
- A **permutation** of the n elements of the set is a *bijective* function $\sigma : [n] \to [n]$.
- Quick question: how many permutations are there on $[n]$?
- The **identity** permutation sends each element to itself.
- A **transposition** is a permutation which exchanges two elements of $[n]$ and leaves all others fixed.
- Permutations can be composed to produce a new permutation.

Two-line and cycle notation

- A permutation can be written in **two-line notation**:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$$

- The permutation above sends $1 \to 5$, 2 is fixed, $3 \to 4$, $4 \to 1$ and $5 \to 3$.
- This is what we'd call a 4-cycle, as we note that $1 \to 5 \to 3 \to 4 \to 1$.
- In **cycle notatation** this is written $(1\ 5\ 3\ 4)$.
- A transposition is a 2-cycle. We denote by s_i the transposition $(i\ i+1)$
- Any permutation can be expressed as a composition of the s_i.

The Symmetric group

- The **Symmetric group**, denoted by S_n, consists of the set of all permutations of $[n]$, with group operation given by function composition, but from now on we will refer to it as multiplication.
- Note that we read composition of functions from right to left.
- Multiplication of permutations in general is *not* commutative. This is worth exploring with students, who have limited experience of non-commutative operations.
- The symmetric group is really important object of study, and has wide-ranging applications throughout algebra.

Conjugacy classes in S_n

- Two elements a and b of a group G are **conjugate** if there exists an element $g \in G$ such that

$$gag^{-1} = b.$$

- Conjugacy is an equivalence relation, and therefore partitions a group into equivalence classes, called **conjugacy classes**.
- The conjugacy classes of S_n are given by **cycle type**. The cycle types of S_n are in one-to-one correspondence with the partitions of n.
- For example, the partition $(3, 2)$ of 5 corresponds to the product of a 3-cycle, with a disjoint 2-cycle. The permutation $(1\ 3\ 2)(4\ 5)$ is an example of such an element.
- It is an interesting exercise for students to confirm/explore this fact.

Symmetric polynomials

- A polynomial in the n variables $x_1, x_2, \ldots, x_n$ is **symmetric** if any permutation of the variables yields the same polynomial.
- For example, in the three variables x, y, and z, the following are symmetric:
  - $x + y + z$.
  - $xy + xz + yz$.
  - $x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2$.
- Any symmetric polynomial in n variables can be expressed as a linear combination of the **monomial symmetric polynomials**, which correspond to integer partitions of length no more than n.

Monomial symmetric polynomials

- Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $r \leq n$.
- If $r < n$, fill out λ with $n - r$ trailing zeroes.
- We denote by x^λ the term $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r}$.
- Obviously, for any variable x_i whose index is greater than r, its power in the product above is 0.
- The monomial symmetric function, m_λ is then defined as the sum over all terms x^α, where α is a permutation of λ. (Don't forget the trailing zeroes.)
- The three examples on the previous slide were all monomial symmetric functions, we had
  - $m_{(1,0,0)}$,
  - $m_{(1,1,0)}$, and
  - $m_{(2,1,0)}$.

Other special symmetric polynomials

- The rth **elementary** symmetric polynomial is the monomial symmetric polynomial corresponding to the partition consisting of r 1s. For example, in three variables:
  - $e_1 = x + y + z$,
  - $e_2 = xy + xz + yz$, and
  - $e_3 = xyz$.
- The rth **complete** symmetric polynomial is the symmetric polynomial consisting of all terms of degree r. For example, in three variables:
  - $h_1 = x + y + z$,
  - $h_2 = x^2 + y^2 + z^2 + xy + xz + yz$, and
  - $h_3 = x^3 + y^3 + z^3 + x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2 + xyz$.
- The rth **power sum** (in three variables) is

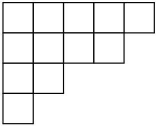
$$x^r + y^r + z^r.$$

Some interesting questions for students

- Best to stick to two variables, still quite a rich space to explore.
  1. Show that h_2 can be written in terms of e_1 and e_2.
  2. Write e_2 in terms of h_1 and h_2.
  3. Use binomial theorem to write an expression for p_3 or p_4 in terms of e_1 and e_2.
  4. Find an expression for p_r in terms of e_1, e_2, p_{r-1} and p_{r-2}.
  5. Explain why any symmetric function can be expressed in terms of e_1, and e_2.
- There are several variations on this theme.

Young diagrams

- We can visualise partitions in pictures, using a left-aligned array of boxes. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, the ith row of a Young diagram contains λ_i boxes.
- The Young diagram of the partition $(5, 4, 2, 1)$ is



- The **conjugate** λ' of a partition λ is the partition whose ith part is the number of boxes in the ith *column* of the Young diagram of λ.
- The conjugate of $(5, 4, 2, 1)$ is $(4, 3, 2, 2, 1)$.
- Why is the conjugate a partition of the same integer?
- Which integers have no self-conjugate partitions? (i.e. $\lambda = \lambda'$.)

Young tableaux

- We can fill the boxes of a Young diagram with the elements of an alphabet, usually taken to be a totally ordered set. For n boxes, usually the set $\{1, 2, \ldots, n\}$.
- A filling of a Young diagram where the entries along the rows are weakly increasing, and down the columns strongly increasing is called **semi-standard**. If we further impose the strictness condition to rows, then the tableau is called **standard**.
- The 7 semi-standard tableaux for the partition $(2, 1)$ are below:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

- A good question for students is to count the number of standard tableaux for a given partition (not too small, and not too large).