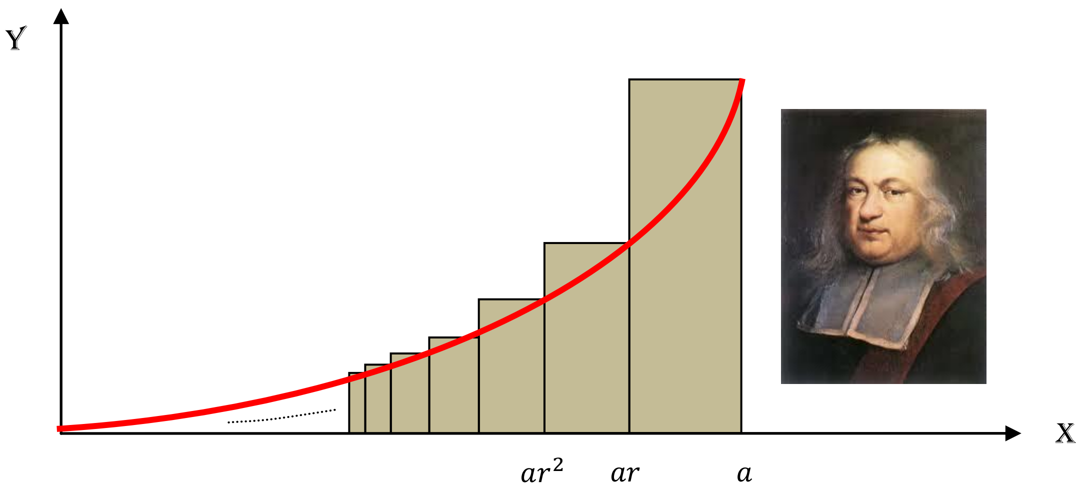
Snapshots on Authentic Concept Development in the Upper Secondary Classroom

(Ed Staples – CMA conference 16 August 2014)



CMA Conference Session 16 August 2014 – Catering for Diversity

Concept development in Mathematics - a few examples:

As a retiring teacher, I thought I might jot down a few of my own ideas about the teaching of mathematics in the secondary school environment. The following three themes are included simply to provide a sense of my approach to concept development in the classroom. After many years of experimentation and reflection, I have come to the view that good teaching always contains a narrative. It is the narrative of human endeavour that resonates within us and holds our attention. Good teaching allows us to authentically sense that endeavour - the historical and cultural context, the people and their ideas, their enquiry and their field testing, the formulation of laws and invention and the triumph of proof and practical application. Good teachers are readers and researchers, building on their knowledge and understanding of those human pursuits. Books like *17 equations that changed the world* by Ian Stewart, *An Imaginary tale, the story of* $\sqrt{-1}$ by Paul Nahin, and the wonderful 3rd edition of *A history of Mathematics – an introduction*, by Victor Katz.

If you are a young mathematics teacher wanting to make a real difference, my advice to you is to always keep reading, get to as many mathematics conferences as you can, get students *doing* things in and outside of the classroom, and spent lots of time building concepts authentically through the narrative of human endeavour.

The examples I share in this paper are drawn from my readings. A few colourful historical variations of the *solution of the quadratic equation*, Henry Briggs invention of the *common logarithms* in 1620, and Fermat's close encounter with infinitesimals in his pursuit of *finding the area under the curve* $y = x^2$.

The number of ways to solve a quadratic equation seems endless, and the various methods shown are worthy of further investigation. *Completing the square* for example was a geometric, rather than an algebraic idea, as was many of the methods invented by Renee Descartes. The geometry of a solution provides fertile ground for algebraic investigation – making connections between the two leads to deep understandings.

Henry Briggs realised that a common logarithm of a number was nothing more than a kind of *cipher text* on a positive real number N. His invention had the additional *conciseness* property that for any integer n , $10^n \times N$ gave rise to encryptions with the same decimal part.

Finally, Pierre de Fermat came perilously close to stumbling on the calculus when he used circumscribed rectangles of *diminishing geometrical width* to determine the area under a quadratic curve. Such a technique is a beautiful example for the classroom, particularly when discussions around limiting geometric sums precede those of integral calculus.

Put together rather hurriedly, please forgive the inconsistency of formatting – so long as you can reflect on the ideas, do a little research and try some of these things in the classroom, I'll feel the paper hasn't been a waste of time.

Ed Staples

August 16 2014

Solving the Quadratic Equation

Solving the quadratic equation is one of the most familiar activities in high school mathematics. The methods of solution date back to antiquity. Indeed the Babylonians 4000 years ago were solving problems that lead naturally to quadratic equations. The Arabic algebraist Muhammed ibn-Musa al-khwarizmi in the 9th century AD was able to show that many problems emerging from inheritances, law suits, trade, land measurement and canal building gave rise to quadratic equations.

1. Al-Khwarizmi: $x^2 + 12x - 108 = 0$

Draw four rectangles of length x and width 3 units surrounding a square of side x as shown in diagram 1.

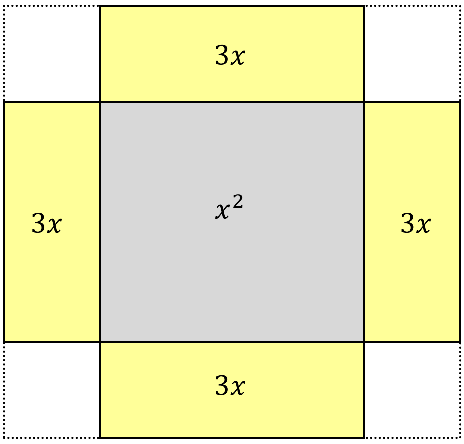


Diagram 1

The area of the square and four rectangles represents $x^2 + 12x$ so the four corners of area $4 \times 9 = 36$ *complete* the larger square. That is, $x^2 + 12x + 36$ is made a *perfect square*, by the addition of 36. So $x^2 + 12x - 108 = 0$ is restated as $(x^2 + 12x + 36) - 144 = 0$ and in accordance with the diagram $(x + 6)^2 - 144 = 0$ which can be easily solved for the positive root $x = 6$. Note that the negative root -18 was considered a false root.

2. Mesopotamian scribes 2150 BCE: $2x^2 + 7x - 4 = 0$

First multiply both sides by the coefficient of x^2

$$(2x)^2 + 7(2x) - 8 = 0$$

Now treat this new form as a *monic* quadratic equation in $2x$ so that $(2x + 8)(2x - 1) = 0$ and thus $x = -4 \; or \; \frac{1}{2}$.

3. Niels Abel (1802 – 1829) $x^2 + 12x - 108 = 0$

Niels Abel was the first mathematician to prove the insolvability of the quintic equation in radicals. His proof contained the result that the root of every quadratic equation with real coefficients can be shown to have the form $x = p + \sqrt{R}$ where p and R are real numbers that can be determined.

Substitute $x = p + \sqrt{R}$ into $x^2 + 12x - 108 = 0$ so that $\left(p + \sqrt{R}\right)^2 + 12\left(p + \sqrt{R}\right) - 108 = 0$ and rearrange to:

$$(p^2 + 12p + R - 108) + (2p + 12)\sqrt{R} = 0$$

Abel shows that for this equation to be true, the two expressions in brackets must *separately* be zero. Thus, from the second expression, $p = -6$ and therefore, substituting into the first bracket, we have $36 + 72 + R - 108 = 0$, and so $R = 144$. This means that $x = p + \sqrt{R} = 6$ as the positive root. Note that the other root is given by $x = p - \sqrt{R} = -18$.

4. Renee Descartes (1596-1650): Form 1: $x^2 = ax + b^2$ and Form 2: $x^2 = ax - b^2$

A twist to the tale of solving quadratic equations is provided by Renee Descartes who examined geometric (and thus positive) solutions to equations of the forms $x^2 = ax + b^2$ and $x^2 = ax - b^2$ where a and b are both positive.

As an example of case 1, solve $x^2 = 2x + 9$ with $a = 2$ and $b = 3$. Take a piece of paper and draw a horizontal line segment AP of length $b = 3$ (see diagram 2). Then from A, erect the perpendicular AO of length $\frac{1}{2}a = 1$. With O as centre, construct a circle of radius $\frac{1}{2}a = 1$. Finally draw in the line through P and O cutting the circle at Q. The length of PQ is the positive root of $x^2 = 2x + 9$.

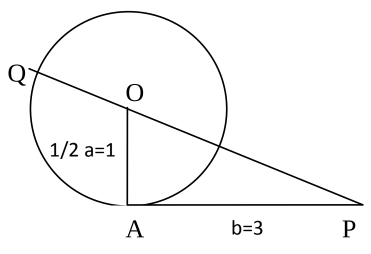


Diagram 2

In the second form, consider the example $x^2 = 10x - 16$. Proceed as before, by constructing AP = 4 ($b = \sqrt{16}$), and erect the perpendicular AO length $\frac{1}{2}a = 5$ and draw the circle as before as shown in Diagram 3

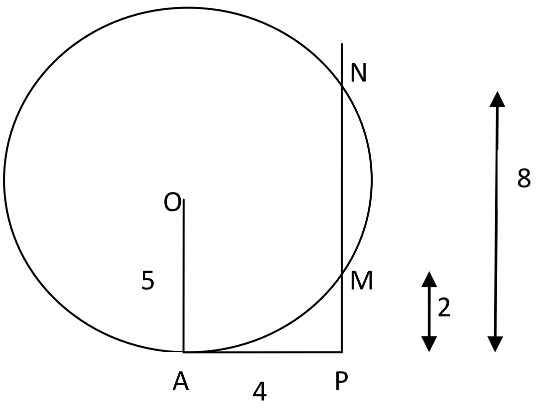


Diagram 3

Erect a perpendicular from P to cut the circle at M and N as shown (if this perpendicular, sufficiently extended doesn't cut the circle, there are no real roots. The length of PM and PN are the roots.

5. An odd quadratic formula: $x^2 + 12x - 108 = 0$

By substitution into $x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$

the roots are $x = \frac{-216}{-12 \mp \sqrt{144+432}} = -6{,}18$

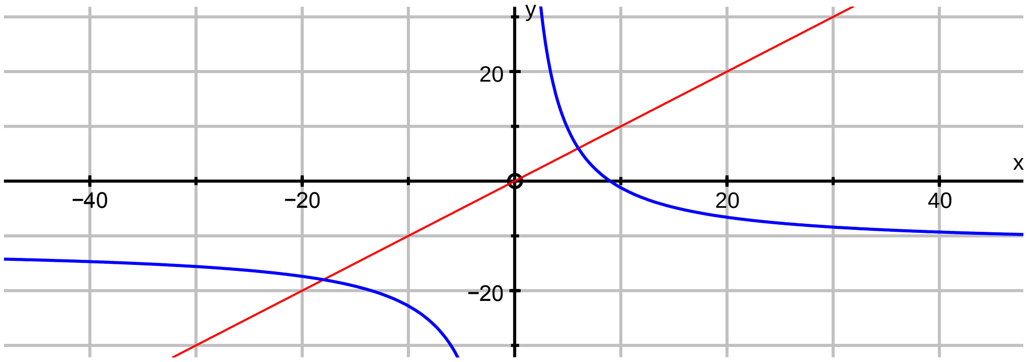
5. Ehrenfried Tschirnhaus (1651 – 1708): $x^2 + 12x - 108 = 0$

The Tschirnhaus transformation is used to eliminate the 2nd highest degree term from any monic polynomial equation $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ by the simple substitution $x = y - \frac{a_{n-1}}{n}$.

In $x^2 + 12x - 108 = 0$ substitute $x = y - \frac{12}{2}$ to get $(y-6)^2 + 12(y-6) - 108 = 0$ and then expanding to $y^2 - 12x + 36 + 12y - 72 - 108 = 0$ and then simplifying to $y^2 = 144$ or $y = \mp 12$. Then $x = y - 6 = 6, or - 18$.

6. An Iterative Technique: $x^2 + 12x - 108 = 0$

From $x^2 + 12x - 108 = 0$ we can rearrange to $= -12 + \frac{108}{x}$. Interpreted graphically, this is the intersection of the line $y = x$ with the hyperbola $y = -12 + \frac{108}{x}$ as shown in diagram 4



We could solve this graphically, by guessing a value close to the root, say $x = 5$, so that $y = -12 + \frac{108}{5} = 9.6$ and then use this as our new estimate of x. Repeating the procedure with the new value we have $y = -12 + \frac{108}{9.6} = -0.75$. On an Excel spreadsheet we continue iterating:

1st Guess	5	9.6
	9.6	-0.75
	-0.75	-156
	-156	-12.6923
	-12.6923	-20.5091
	-20.5091	-17.266
	-17.266	-18.2551
	-18.2551	-17.9162
	-17.9162	-18.0281
	-18.0281	-17.9907
	-17.9907	-18.0031
	-18.0031	-17.999
	-17.999	-18.0003
	-18.0003	-17.9999
	-17.9999	-18
Root	-18	-18

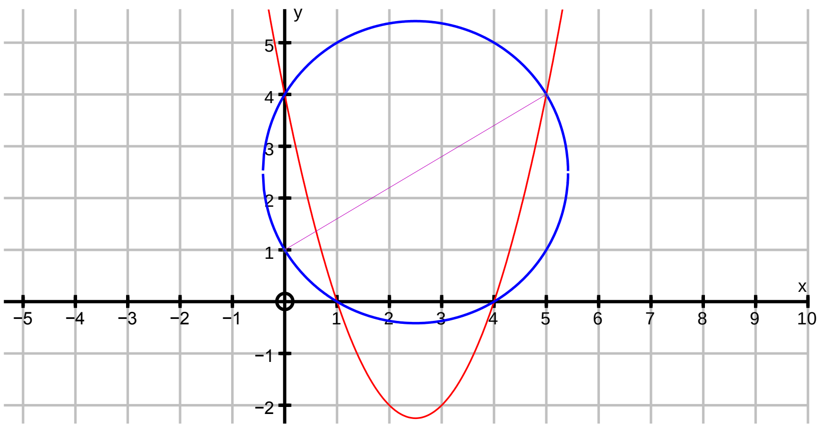
Depending on the value of the coefficients, this technique will always deliver one of the roots (in this case $= -18$).

7. Finding quadratic roots geometrically: $x^2 + bx + c = 0$

In days gone by, perhaps before graphic calculators and computer software were easily accessible, an ingenious method was developed to allow anyone so interested to determine graphically the real roots of any quadratic equation. The method uses the fact that a circle is far easier to draw than a parabola, provided of course you have the right equipment.

Here is the method for the example $x^2 - 5x + 4 = 0$:

The coefficient of x is -5 and the constant term is 4, so mark carefully on the Cartesian plane the points $(0,1) and\ Q(-b, c) = (5,4)$. Draw a line segment between them. Find the centre and the length of that line segment using the mid-point and distance formula. In our example, the centre is $\left(\frac{5}{2}, \frac{5}{2}\right)$ and $d = \sqrt{24} \approx 4.9$. Draw a circle centre $\left(\frac{5}{2}, \frac{5}{2}\right)$ and radius $\frac{d}{2} = 2.45$ and read off where this circle cuts the x axis. These are the roots.



Graph 6

Why does it work?

A circle's general equation can be written $x^2 + y^2 + bx + my + c = 0$. This can be rearranged to $(x^2 + bx + c) + (y^2 + my) = 0$.

Thus, the circle's roots (putting $y = 0$ into its equation) will be precisely the same as the quadratic roots $x^2 + bx + c = 0$. We also know that the circle passes through $P(0,1)$ and thus it is true that $y^2 + my + c = 0$. This last equation has the root $y = 1$ and thus can be re-expressed as $(y - 1)(y - t) = 0$ for some unknown value t. Expanding shows $y^2 - (t + 1)y + t = 0$ and thus $t = c$ and $m = -(c + 1)$. Thus our circle has the equation $(x^2 + bx + c) + y^2 - (c + 1)y = 0$, and by direct substitution, we can show that the point $Q(-b, c)$ is on it.

Henry Briggs and his Common Logarithms

John Napier logarithms were published in 1620 (d.1617). It took 20 years to compile the hand calculated numbers. Henry Briggs was fascinated by Napier's approach and in 1624 published his *common logarithms*, which were a huge improvement on the idea. Both men's objective was to simplify mathematical calculations - in particular to find a way to turn problems involving multiplication and division into ones of addition and subtraction.

Quarter square multiplication

Much earlier, around 4000BC, the Babylonians had discovered a method to simplify multiplication using look-up tables, and perhaps this is a good starting point for our discussion. We'll take a simple example and use modern notation to illustrate the method.

Suppose we wish to multiply 4 by 7. We first find the positive sum and difference of 4 and 7 which is obviously 3 and 11. Then we simply look up the "codes" for 3 and 11 in the look-up table shown in table 1:

Number	Code	Number	Code	Number	Code	Number	Code
1	0	26	169	51	650	76	1444
2	1	27	182	52	676	77	1482
3	2	28	196	53	702	78	1521
4	4	29	210	54	729	79	1560
5	6	30	225	55	756	80	1600
6	9	31	240	56	784	81	1640
7	12	32	256	57	812	82	1681
8	16	33	272	58	841	83	1722
9	20	34	289	59	870	84	1764
10	25	35	306	60	900	85	1806
11	30	36	324	61	930	86	1849
12	36	37	342	62	961	87	1892
13	42	38	361	63	992	88	1936
14	49	39	380	64	1024	89	1980
15	56	40	400	65	1056	90	2025
16	64	41	420	66	1089	91	2070
17	72	42	441	67	1122	92	2116
18	81	43	462	68	1156	93	2162
19	90	44	484	69	1190	94	2209
20	100	45	506	70	1225	95	2256
21	110	46	529	71	1260	96	2304
22	121	47	552	72	1296	97	2352
23	132	48	576	73	1332	98	2401
24	144	49	600	74	1369	99	2450
25	156	50	625	75	1406	100	2500

Table 1: Quarter square table

The code for 3 is *2* and the code for 11 is *30*. We then simply subtract the smaller code from the larger code to reveal the answer as *28*.

Try a couple to convince yourself that the method actually works. Also what about 17^2?

So why does it work?

In modern notation, the Babylonians knew that $xy = \left\lfloor \frac{(x+y)^2}{4} \right\rfloor - \left\lfloor \frac{(x-y)^2}{4} \right\rfloor$ where the strange bracket used is called the *floor function* which simply means that for positive numbers, delete the fractional part of the result (for example $\lfloor 3.25 \rfloor = 3$).

So in our example, the code for 11 was $\left\lfloor \frac{11^2}{4} \right\rfloor = 30$ and code for 3 was $\left\lfloor \frac{3^2}{4} \right\rfloor = 2$, and the subtraction $30 - 2$ yielded the answer.

Common Logs

If we now turn to the work of Henry Briggs, he too invented a look-up table, similar to table 2.

Number	Log	Number	Log	Number	Log	Number	Log	Number	Log
1	0.0000	3	0.4771	5	0.6990	7	0.8451	9	0.9542
1.1	0.0414	3.1	0.4914	5.1	0.7076	7.1	0.8513	9.1	0.9590
1.2	0.0792	3.2	0.5051	5.2	0.7160	7.2	0.8573	9.2	0.9638
1.3	0.1139	3.3	0.5185	5.3	0.7243	7.3	0.8633	9.3	0.9685
1.4	0.1461	3.4	0.5315	5.4	0.7324	7.4	0.8692	9.4	0.9731
1.5	0.1761	3.5	0.5441	5.5	0.7404	7.5	0.8751	9.5	0.9777
1.6	0.2041	3.6	0.5563	5.6	0.7482	7.6	0.8808	9.6	0.9823
1.7	0.2304	3.7	0.5682	5.7	0.7559	7.7	0.8865	9.7	0.9868
1.8	0.2553	3.8	0.5798	5.8	0.7634	7.8	0.8921	9.8	0.9912
1.9	0.2788	3.9	0.5911	5.9	0.7709	7.9	0.8976	9.9	0.9956
2	0.3010	4	0.6021	6	0.7782	8	0.9031	10	1.0000
2.1	0.3222	4.1	0.6128	6.1	0.7853	8.1	0.9085		
2.2	0.3424	4.2	0.6232	6.2	0.7924	8.2	0.9138		
2.3	0.3617	4.3	0.6335	6.3	0.7993	8.3	0.9191		
2.4	0.3802	4.4	0.6435	6.4	0.8062	8.4	0.9243		
2.5	0.3979	4.5	0.6532	6.5	0.8129	8.5	0.9294		
2.6	0.4150	4.6	0.6628	6.6	0.8195	8.6	0.9345		
2.7	0.4314	4.7	0.6721	6.7	0.8261	8.7	0.9395		
2.8	0.4472	4.8	0.6812	6.8	0.8325	8.8	0.9445		
2.9	0.4624	4.9	0.6902	6.9	0.8388	8.9	0.9494		

Table 2: Briggs Logs

Suppose we wish to multiply 2 by 3 (granted, a trivial example!) Using Briggs method, we look up the "log" of 2 which is 0.3010 and the "log" of 3, which is 0.4771, and add them! This yields 0.7781. Now use the table in reverse – find 0.7781 amongst the logs and look back to the number 6 (the slight discrepancy in my table is due to the rounding of the logs to four decimal places for the purposes of this demonstration).

Taking another example, say 1.4 by 3.5. The sum of the logs 0.1461 and 0.5441 is 0.6902, and looking this up reveals the answer as 4.9.

Briggs secret was indices. In fact his logs were just indices! In the first example, Briggs had worked out that $2 = 10^{0.3010}$ and $3 = 10^{0.4771}$ and so:

$$2 \times 3 = 10^{0.3010} \times 10^{0.4771} = 10^{0.3010+0.4771} = 10^{0.7781}.$$

A multiplication problem has been turned into an addition problem. To turn $10^{0.7781}$ back into an ordinary number required a table of anti-logs, but we were able to find the number using table 2.

Using a "base" of 10, Briggs found other advantages. Suppose we consider the product 14×35. Briggs knew that the logs could be determined from Table 2, even though it only goes to 10. This is because 14 can be written as 1.4×10^1 and $35 = 3.5 \times 10^1$.

This means: $14 = 10^{0.1461} \times 10^1 = 10^{1.1461}$

And: $35 = 10^{0.5441} \times 10^1 = 10^{1.5441}$

What Briggs had discovered was that he really only need a log table for numbers from 1 to 10 (although more decimal places were required). For example the logarithm for 2400 would be 3.3802 from the table. He called the 3, the *characteristic* of the log, and he called the decimal part 0.3802 the *mantissa* of the log (Latin for "makeweight" which was a small weight used to balance weights on weighing scales).

Division of two numbers meant the *subtraction* of the logs, and *squaring* the number meant *doubling* the logs.

Common logarithms are base 10 logarithms, but we are not confined to just base 10. For example we could say that because $2^5 = 32$ then the logarithm of 32, using base 2, is 5. A shorthand way of writing this is $\log_2 32 = 5$.

In general, if $\log_b a = c$ then $a = b^c$ and *these two equations are really saying the same thing in two different ways.*

Today the study of logarithms is an important area of mathematics and now includes the study of log functions. There are a wide range of applications where log theory is needed including the measurement of sound, and earthquakes, and the study of relationships between certain physical phenomena.

How Briggs worked out his 20,000 Logarithms

It took Briggs three years of hard work to determine his logarithms. He worked on primes and then constructed the composites by adding the relevant prime logs together. As an example of his method, let's think about his approach for log 2.

He started by knowing that $2^{10} = 1024$ and that 1024 is expressible in exponents as$10^{3.something}$. Hence he knew that $3 < \log_{10} 1024 < 4$. This meant that $3 < \log_{10} 2^{10} < 4$ and therefore that

$3 < 10 \cdot \log_{10} 2 < 4$, and this means that $\frac{3}{10} < log(2) < \frac{4}{10}$. In other words $\log_{10} 2$ is somewhere between 0.3 and 0.4 .

To get more accuracy $2^{20} = 1{,}048{,}576$ and so $6 < \log_{10} 2^{20} < 7$ and so $\log_{10} 2$ is somewhere between 0.3 and 0.35. Higher and higher powers of 2 deliver more accurate estimates. Briggs continued so that each prime from 1 to 20,000 was accurate to 14 decimal places.

Fermat's technique – a twist on the story

In a previous article I examined a left field introduction to finding areas under curves – a fairly standard lead up to integration. I'll briefly describe it again.

For many years in my teaching, I began my discussion of areas under curves by determining an approximate value for the area under the curve $y = x^2$ between $x = 0$ and say $x = a$. Sometime later, I discovered the Pierre de Fermat sometime after 1629 had used geometric sequences and sums to arrive at an answer. Knowing that the topic sequences and series preceded calculus, I wondered why this approach was generally considered in the normal course of events. So here is the method. Break up the area into circumscribed rectangles of diminishing width, by starting your first rectangle off with its right hand vertical edge at $x = a$ (the idea is shown in Diagram 1). The second rectangle, to the left, has its right hand edge at $x = ar$ where r is a positive geometric ratio less than 1. This means that the second rectangle will be slightly narrower than the first. The third rectangle's right hand edge is then drawn at $x = ar^2$ and the fourth rectangle's edge is at $x = ar^3$ and this continues on and on with each new rectangle narrower than the one that preceded it. Note that no finite number of rectangles will ever breach the origin if we continue in this manner.

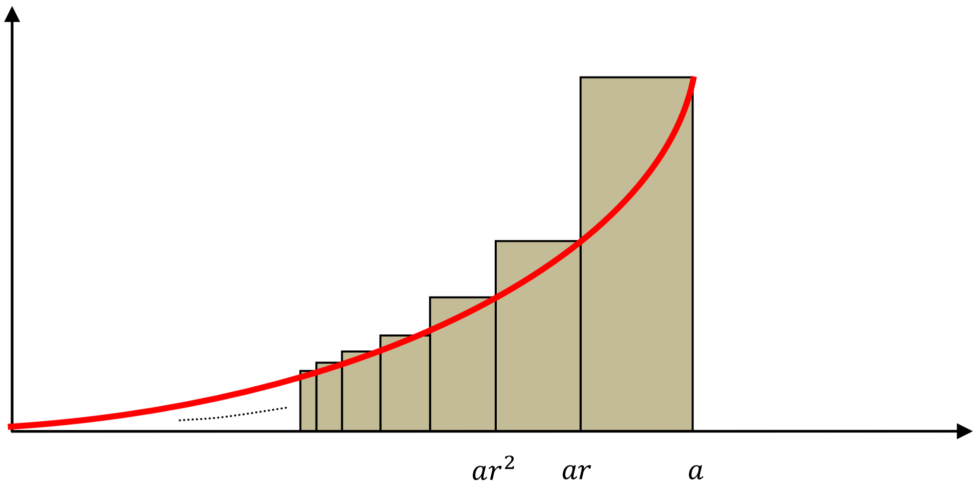


Diagram 1: The diminishing width rectangles on $y = x^2$

We now simply "add up" the areas of these rectangles using the limiting sum of a Geometric Progression as follows:

Total Rectangle Area = $(a - ar) \cdot a^2 + (ar - ar^2) \cdot (ar)^2 + (ar^2 - ar^3) \cdot (ar^2)^2 + \cdots$

Taking out $a^3(1 - r)$ as a common factor we have:

Total Rectangle Area =$a^3(1 - r)\{1 + r^3 + r^6 + \cdots\}$

The expression in parentheses sums geometrically to $\frac{1}{1-r^3} = \frac{1}{(1-r)(1+r+r^2)}$ and thus, after cancelling we arrive at:

Total Rectangle Area = $\frac{a^3}{(1+r+r^2)}$

What a result! Before us we see the most beautiful expression, for if we let r move toward 1 (implying an infinitely large number of infinitely small width rectangles), in the limit, the total rectangle area is none other than $\frac{a^3}{3}$.